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AN OPTIMAL STRATEGY FOR TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY COMPUTER

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ABSTRACT

The topological approach to the analysis of bilateral networks has been important for many years because, for certain classes of networks, it is computationally very efficient. Within the last ten years many investigators have conceived and developed extensions of the basic theory, separate methods to analyze general linear networks which may contain unilateral elements. One might well ask, "Why so many separate methods?". The answer lies in the complexity of the problem which offers many possible avenues of development. The main substance of this paper is a new method which is comparatively simple and well suited to the analysis of general networks by computer.

The first part of the paper discusses many of the important contributions of the previous investigators. An analysis of the procedures and selected examples for some of the methods are given.

The second part of the paper shows the development of a new method which is optimal in some senses. A mathematical development based on several new topological theorems shows how network functions can be derived from the directed trees and two-trees of a partly oriented graph. It clarifies the relationships between nonoriented, oriented, and partly-oriented graphs and shows that some earlier methods using oriented graphs overspecify the network with redundant information for each bilateral element.

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INTRODUCTION

Topological methods of network analysis have had a long history of development from the time of Kirchhoff and Maxwell. Many investigators in the academic community have been very faithful about reporting contributions of predecessors and contemporaries in the field and have generally contributed to the large fund of knowledge presently available on the subject. Men such as Percival [2], Coates [4], Mason [3], Mayeda [1], Seshu [10], and many others have each made some important contributions and when taken together their efforts in topological analysis constitute a definitive body of knowledge on the subject. So many different and in some aspects redundant methods have been introduced that a re-evaluation and a survey of these methods according to the standards that exist in engineering practice today should be of great value. One objective of this paper is to attempt such a re-evaluation.

It is certainly not possible to say what Kirchhoff and Maxwell held as a prime motivation for instituting the topological analysis technique but it is possible to say that for most practitioners of the art today, the ability to analyze small electrical networks by hand and the ability to formulate efficient computer analysis routines for intermediate size networks represents the consensus. The efficiency and the simplicity of the topological technique can hardly be disputed when compared with certain other methods of analysis though this fact might easily be obscured in the process of reading through some of the mathematical proofs and derivations presented by the numerous investigators.

Because of this promise of simplicity and efficiency in hand calculation and the possibility of improving on the state-of-the-art in computer aided topological analysis, the purpose of this paper is to review the important recent contributions to the art and present a new and improved method.

Fundamental Characteristics of the Topological Method.

The topological method is basically a symbolic method and can be compared with numerical methods such as the state-space and matrix methods based on Kirchhoff's voltage and current equations. Because this is true, none of the problems of numerical stability or accumulation of error found in the numerical method are encountered until an actual numerical result is required. A network can be analyzed symbolically and a numerical result can be derived from a symbolic solution by assigning numerical values at the end of the analysis process. One can, by these means, obtain any desired degree of accuracy. This may seem at first to be a trivial recommendation, however in practice, this problem of accuracy and numerical stability has become the dominant problem. There is a price to be paid for symbolic analysis however. It shows up clearly when a network of approximately fifteen nodes and twenty branches is analyzed in computer. For such a network there is a possibility of obtaining over 150,000 terms in the expansion of the determinant of the node admittance matrix. Clearly, such a network is too large to deal with by hand and the value of knowing symbolicly the relationships between 20 network elements is questionable. Since the complexity of symbolic analysis grows disproportionately with increasing network size, a finite bound exists on the application of this technique even when the computer is used and numerical values are substituted into the symbolic analysis results to produce an answer.

For the larger networks, the greatest promise seems to be offered by the state-space technique if the numerical difficulties can be overcome.

In computing the frequency characteristics of a network by state-space analysis, it is necessary to evaluate the function (sI-A) where s is the complex frequency, A is the state-space parameter matrix characterizing the network, and I is the identity matrix. This is essentially a symbolic matrix in s with numerical coefficients if A is entirely numerical. Procedures such as the Faddeev Frame Souriau algorithm [16] have been used but with difficulty because unexpected errors can easily accumulate and cause trouble. Recent experiences with the QR algorithm of Francis [22] [23] indicate that better results are possible. However, there are still problems especially for networks with combinations of small and large time constants. Networks of this type cause the eigenvalues of the (sI-A) matrix (natural resonances of the network) to vary over large numerical limits thus precipitating the problems peculiar to the "Ill Behaved Matrix". Signal flow graph methods are another topological technique for network analysis and, as shown by Mason [17] and Coates [19] can be considered as a subset of the methods discussed here. Happ, Carpenter [20] and others have used flow graph techniques in computer aided analysis and one cannot completely discount the possibility of important developments in this field. Pritsker [21] has also used a graph reduction technique and has shown the value of this technique in evaluating weighted schedule diagrams. But we will not discuss flow graph methods, per se, further in this paper.

Organization of the Paper.

The major content of this paper has been divided into three parts. The first part constitutes a review of a number of recent papers on topological analysis. Each of these papers will be discussed individually according to the following format: A short outline of the procedure for obtaining the determinant of the node admittance matrix of a general nonreciprocal network will be presented. This will be accompanied in each case by discussion of the recommending features and the weak points as they apply to hand and computer aided analysis. Examples of the procedures will be presented.

The second major part of the paper will be a comparative discussion of all the methods that were reviewed in the first part and an analysis of the common features among all methods. A serious attempt to obtain an objective comparison will be made, however, some of the arguments presented might justifiably be considered arbitrary from the point of view of those with different objectives in mind.

In the third part of this paper a new method of topological analysis of general networks will be presented. Here an attempt is made to restructure and simplify earlier techniques showing the very close relationship between topological analysis for general networks and that for reciprocal networks. An outline of a proof for this technique is presented and appropriate examples are worked to illustrate the method.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF D. P. BROWN.

In the development of ordinary topological analysis of bilateral networks, a relationship is made between the terms in the determinant of the node admittance matrix and the trees of the graph associated with the network. This relationship demands that the network edge admittance matrix be diagonal. In the analysis of general networks, the edge admittance matrix is not necessarily diagonal. In order to rectify this unfortunate fact of life, Brown has devised a way to model all non-reciprocal elements by a 2 x 2 submatrix of admittances. This sub-matrix is then treated as a single admittance and the ordinary rules for finding network functions by finding trees of the network graph are used. It is then necessary to expand the result thus obtained by the determinants of the matrices representing the coupled elements. Because the algebraic modeling process is somewhat complex and the rules for determining the admittance products are quite involved, the method seems very cumbersome. The following is a step by step procedure of the method.

Preliminary Limitation.

No coupled elements which produce a singular 2 x 2 admittance matrix are allowed.

Procedure.

I. Describe all coupled elements by means of a 2 x 2 matrix relating the voltage and current in these elements.

II. Assign voltage directions across each element.

III. Arrange an oriented graph G of all elements. It is necessary to retain edge orientation even for passive elements.

IV. Find all trees of G.

V. Inspect all the trees and compute tree admittance products according to sub-procedure 1 for those trees which do not contain any coupled elements (one of two elements expressed by the 2 x 2 matrices computed in I above).

Sub-procedure 1.

This procedure identifies all trees containing only bilateral or uncoupled elements and treats them as a separate set. Tree admittance products are computed for every member of this set in the ordinary manner. For example, if the following are trees containing only bilateral elements: $\{1, 2, 3; 1, 2, 4; 1, 3, 4\}$ then the tree admittance parameters are: $\{Y_1 Y_2 Y_3, Y_1 Y_2 Y_4, Y_1 Y_3 Y_4\}$.

VI. For those trees which contain coupled elements, inspect and separate out all those that contain both elements of a coupled set and compute the tree admittance products according to sub-procedure 2.

Sub-procedure 2.

This procedure identifies all trees containing both elements of a coupled pair and treats them as a separate set. Tree admittance products are computed for every member of this set by first computing the determinant of the admittance matrix Y_A representing the volt-amp relationships between the coupled elements, then, using the result as a factor representing the pair, compute the required admittance product. For example, let elements a and b be coupled elements. Further, assume that a tree 1, 2, a, b, 3 exists. The determinant of the admittance matrix representing the coupled pair is $Y_{aa} Y_{bb} - Y_{ab} Y_{ba}$ and the tree admittance product is: $Y_1 Y_2 Y_3 Y_{aa} Y_{bb} - Y_1 Y_2 Y_3 Y_{ab} Y_{ba}$.

VII. For those trees which contain only one element of a coupled set or groups of elements of coupled sets, the counterparts of which are not in the tree, compute tree admittance products according to sub-procedure 3.

Sub-procedure 3.

This procedure identifies all trees containing one element A of a coupled pair or coupled pairs a_p, b_p. The other element or elements b_p which are not in the tree must therefore be chords or links. Tree admittance products are computed for every member of the set by first computing the determinant of a special matrix Y_e representing the interrelationships between these elements. Then, using this result as a factor representing the elements, the tree admittance products are computed.

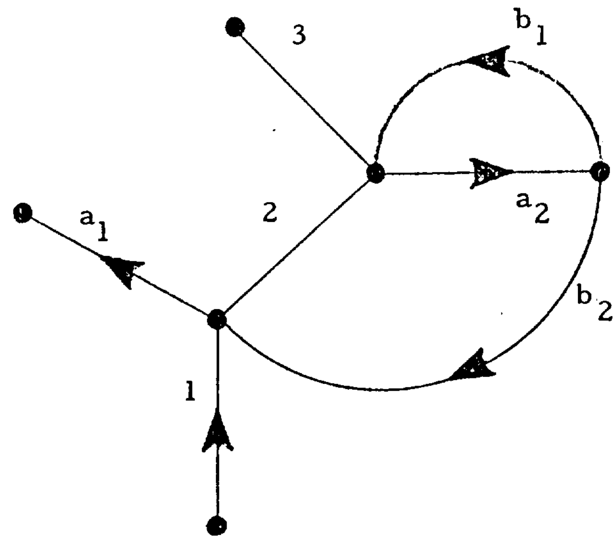
The order of the special matrix is p where p is the number of coupled elements with no counterpart in the tree in question. Diagonal entries in Y_e are found in the following way:

$$Y_{pp} = \begin{cases} Y_{a_p, a_p} & \text{if the pth edge does not lie in the loop completed by } b_p. \\ Y_{a_p, a_p} \pm Y_{b_p, a_p} & \text{if the pth edge lies in the loop completed by link } b_p \text{ and the + sign is taken if } a_p \text{ and } b_p \text{ have the same orientation.} \end{cases}$$

Off diagonal elements of Y_e are found in the following way:

$$Y_{qp} = \begin{cases} \pm Y_{b_p, a_p} & \text{if the edge } a_q \text{ lies in the loop completed by } b_p. \\ 0 & \text{otherwise and the + sign is taken for confluence of } a_q \text{ and } b_p. \end{cases}$$

As an example, consider the tree 1, 2, a_1, 3, a_2. Assuming the following topology for the tree:



$$Y_e = \begin{bmatrix} Y_{a_1} & 0 \\ (Y_{b_1}, Y_{a1}) & (Y_{a_2, a_2} + Y_{b_2, a_2}) \end{bmatrix}$$

and ,

$$\text{Det } Y_2 = (Y_{a_1, a_1})\ (Y_{a_2, a_2} + Y_{b_2, a_2}).$$

Therefore, the tree admittance product computed for this tree is:

$$Y_1\, Y_2\, Y_3\, (Y_{a_1},\, a_1)\, (Y_{a_2},\, a_2)$$
$$+\, Y_1\, Y_2\, Y_3\, (Y_{a_1},\, a_1)\, (Y_{b_2},\, a_2).$$

VIII. For those trees which contain both elements of coupled sets as well as single elements with their counterparts, a combination of sub-procedures 2 and 3 is used. Since the classes discussed in VI and VII above are mutually exclusive, it is enough to multiply Det Y_e of sub-procedure 3 by Det Y_{a_1} Det Y_{a_2} . . . Det Y_{a_r} of sub-procedure 2 and also by the admittances Y_i of the uncoupled elements to obtain tree admittance parameters for trees of this type.

IX. Sum the products developed in V, VI, VII and VIII above to produce the determinant of the node admittance matrix. An example is shown in figure 1.

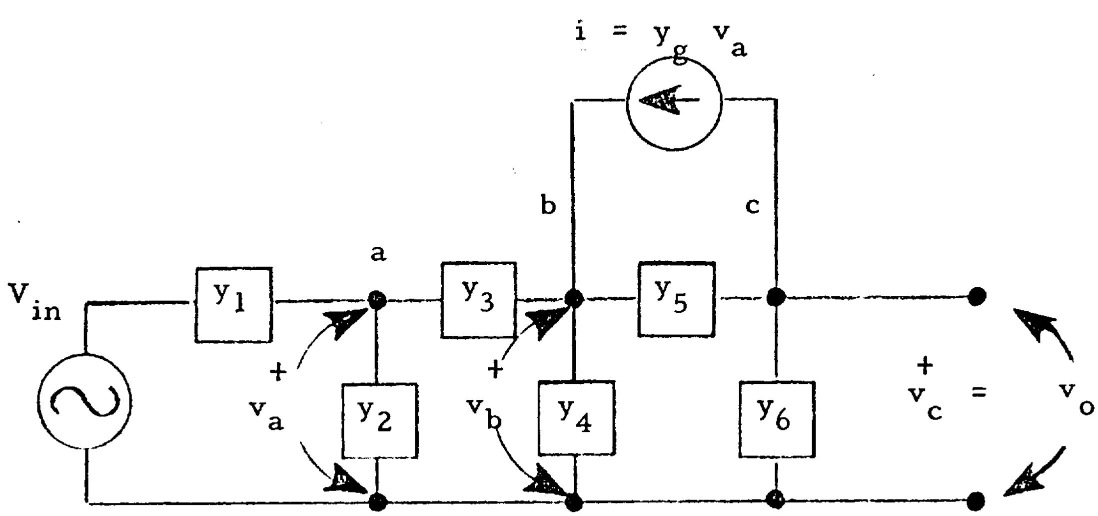


Figure 1.

Example Network.

Our first problem is to find out the nature of the coupled network immeshed within the total network. This cannot be done easily for the general case by inspection and therefore represents a separate computational task which in some ways is irrelevant to the ideal topological approach. Only certain kinds of coupled networks are permissible - those which produce a nonsingular 2 x 2 submatrix within the otherwise diagonal matrix of admittances representing the network. Let us consider the Kirchhoff's current law equations of this network expressed in matrix form as follows:

$$Y_N \; V_N = I_N .$$

Specifically for this network:

$$\begin{bmatrix} (y_1 + y_2 + y_3) & -y_3 & 0 \\ -(y_g + y_3) & (y_3 + y_4 + y_5) & -y_5 \\ y_g & -y_5 & (y_5 + y_6) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} y_1 \; V_{in} \\ 0 \\ 0 \end{bmatrix}.$$

Let us decompose this expression into topological information and admittance parameters by recognizing that:

$$Y_n = A \; Y_e \; A^t$$

where A is the incidence matrix of the graph which represents the network and Y_e is the edge admittance matrix.

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}.$$

If we treat Y_e as an unknown matrix X, with coupled edges represented by entries in columns 2 and 5, we have:

$$Y_e = X = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 5 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} x_{11} & & & & & \\ & x_{22} & x_{25} & & & \\ & x_{52} & x_{55} & & & \\ & & & x_{33} & & \\ & & & & x_{44} & \\ & & & & & x_{66} \end{bmatrix} \end{array}.$$

Rearranging the coupling elements 2 and 5 in the A matrix, we have:

$$A = \begin{array}{c|cccccc} & 1 & 2 & 5 & 3 & 4 & 6 \\ \hline a & 1 & 1 & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 1 & -1 & 1 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 1 \end{array}$$

and,

$$A\,Y_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & & & & & \\ & x_{22} & x_{25} & & & \\ & x_{52} & x_{55} & & & \\ & & & x_{33} & & \\ & & & & x_{44} & \\ & & & & & x_{66} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{22} & x_{25} & x_{33} & 0 & 0 \\ 0 & x_{52} & x_{55} & -x_{33} & x_{44} & 0 \\ 0 & -x_{52} & -x_{55} & 0 & 0 & x_{66} \end{bmatrix}.$$

$$A \; Y_e \; A^t = Y_n$$

$$= \begin{bmatrix} x_{11} & x_{22} & x_{25} & x_{33} & 0 & 0 \\ 0 & x_{52} & x_{55} & -x_{33} & x_{44} & 0 \\ 0 & -x_{52} & -x_{55} & 0 & 0 & x_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}+x_{22}+x_{33} & x_{25}-x_{33} & -x_{25} \\ x_{52}-x_{33} & x_{33}+x_{44}+x_{55} & -x_{55} \\ -x_{52} & -x_{55} & x_{55}+x_{66} \end{bmatrix}.$$

Each term in this matrix can be equated with a term in the Y_n matrix given earlier and the values of the x's are found in terms of the y's.

$$x_{11} = y_1 \qquad x_{44} = y_4 \qquad x_{25} = 0$$

$$x_{22} = y_2 \qquad x_{55} = y_5 \qquad x_{52} = y_g$$

$$x_{33} = y_3 \qquad x_{66} = y_6 .$$

Our coupled group is therefore:

$$Y_c = \begin{bmatrix} y_2 & 0 \\ -y_g & y_5 \end{bmatrix}.$$

$$Y_c \; V_c = I_c,$$

or

$$\begin{bmatrix} y_2 & 0 \\ -y_g & y_5 \end{bmatrix} \begin{bmatrix} v_2 \\ v_5 \end{bmatrix} = \begin{bmatrix} i_2 \\ i_5 \end{bmatrix}.$$

This is the volt-amp relation for a loaded voltage controlled current source such as that shown in figure 2. If the current generator had not been shunted by admittance Y_5, we would have had a singular matrix and the problem could not be solved in this way.

The graph associated with the original network is shown in figure 3.

Computing trees of the graph G, we have:

3 4 5	1 4 5	2 3 6
3 4 6	1 4 6	2 4 5
3 5 6	1 5 6	2 4 6
1 3 5	2 3 5	2 5 6
1 3 6		

The trees encircled are those containing coupled elements.

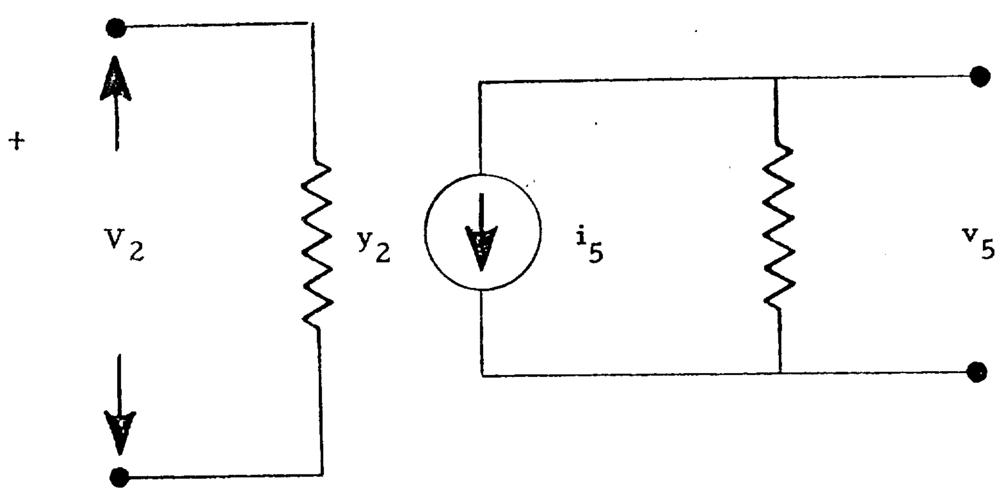


Figure 2.

Voltage Controlled Current Source.

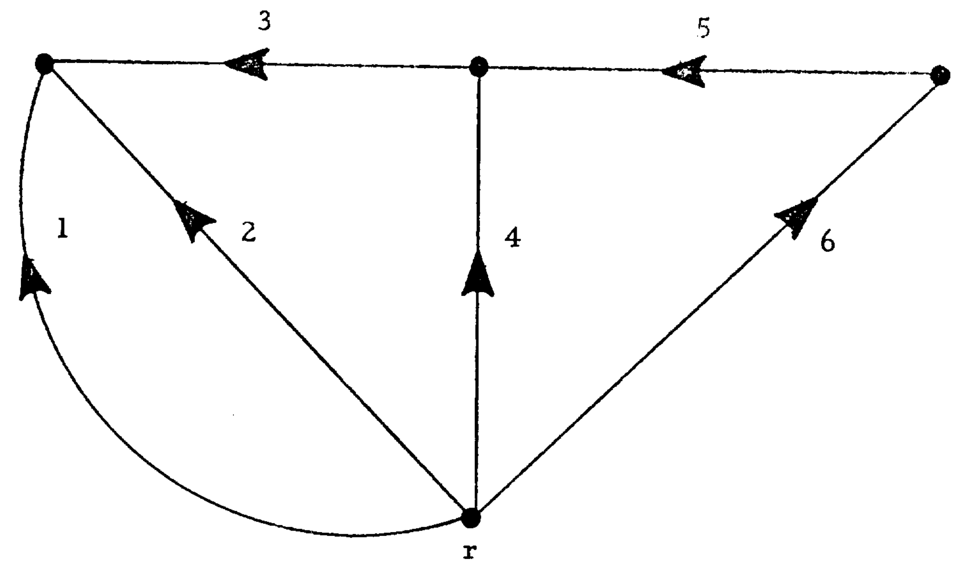


Figure 3.

Graph of the Example Network.

For uncoupled elements, tree admittance products are computed exactly as they would be in ordinary topological analysis of passive networks.

The following is a list of trees without coupling elements associated with their tree admittance products as obtained by subprocedure 1.

$$1 \quad 3 \quad 6 \quad \Longrightarrow \quad y_1 \; y_3 \; y_6$$
$$1 \quad 4 \quad 6 \quad \Longrightarrow \quad y_1 \; y_4 \; y_6$$
$$3 \quad 4 \quad 6 \quad \Longrightarrow \quad y_3 \; y_4 \; y_6 .$$

For trees containing coupling elements, the very complex set of rules used to establish edge weight will be shown.

Tree admittance products obtained by sub-procedure 2 are as follows:

$$2\ 3\ 5 \Longrightarrow y_3 \text{ Det } y_c = y_2\ y_3\ y_5$$
$$2\ 4\ 5 \Longrightarrow y_4 \text{ Det } y_c = y_2\ y_4\ y_5$$
$$2\ 5\ 6 \Longrightarrow y_6 \text{ Det } y_c = y_2\ y_5\ y_6 .$$

Those obtained by sub-procedure 3 are as follows:

$$1\ 3\ 5 \Longrightarrow y_1\ y_3\ y_5$$
$$1\ 4\ 5 \Longrightarrow y_1\ y_4\ y_5$$
$$1\ 5\ 6 \Longrightarrow y_1\ y_5\ y_6$$
$$2\ 3\ 6 \Longrightarrow (y_2 - y_g)\ (y_3\ y_6)$$
$$2\ 4\ 6 \Longrightarrow y_2\ y_4\ y_6$$
$$3\ 4\ 5 \Longrightarrow y_3\ y_4\ y_5$$
$$3\ 5\ 6 \Longrightarrow y_3\ y_5\ y_6 .$$

Therefore:

$$\begin{aligned}\Delta = \ & y_1\ y_3\ y_6 + y_1\ y_4\ y_6 + y_3\ y_4\ y_6 + \\ & + y_2\ y_3\ y_5 + y_2\ y_4\ y_5 + y_2\ y_5\ y_6 + \\ & + y_1\ y_3\ y_5 + y_1\ y_4\ y_5 + y_1\ y_5\ y_6 + \\ & - y_g\ y_2\ y_3 + y_2\ y_3\ y_6 + y_2\ y_4\ y_6 + \\ & + y_3\ y_4\ y_5 + y_3\ y_5\ y_6 .\end{aligned}$$

In order to begin this topological process it is necessary to model non-reciprocal elements by small matrices. This is an involved process to perform by hand and a challenging programming problem if done by computer. Here an algebraic step is injected into the solution and one can question whether it is still valid in the general case to consider the method exclusively topological. It might be more meaningful if the original network topology were modified to ease the burden of subsequent computation. Brown claims the advantage of using original network topology but this is obviously a complicating restraint. Furthermore it is not possible to live within this restraint if a network containing a non-loaded current generator or certain other kinds of active networks are encountered. This fact coupled with the difficulty of obtaining tree admittance products by three separate sub-procedures results in complexity that diminishes the utility of the method. The necessity for forming a fully oriented graph and using edge orientation in sub-procedure 3 in the determination of the admittance products also complicates the method. Most of the difficulty of computing sign terms is eleminated but the price that is paid shows mainly in sub-procedure 3.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF W. K. CHEN.

Chen has approached the problem of network analysis from a somewhat different point of view. He assumes as a point of departure that analysis is being done by hand and that the node admittance matrix or loop impedance matrix is available. This assumption is somewhat limiting in that the algebraic formation of these matrices should, if possible, be avoided if we are to exploit all the benefits of the topological method. However, the method is still very useful under the conditions for which it was intended.

Once the loop impedance or node admittance matrix is formed, Chen has shown that a directed graph representing the matrix can be used to compute determinants and cofactors which, of course, can in turn, be used to compute network functions. These determinants and cofactors are shown to be obtainable from the directed trees and directed 2-trees of the graph.

Procedure.

I. Form the node admittance matrix. (One could proceed in simular manner with loop impedance matrix.)

II. Form the directed graph or digraph G from the entries in the node admittance matrix according to the following rules:

a. There are (n + 1) vertices where n is the order of the node admittance matrix Y_n (vertex r being the reference vertex).

b. Identify each diagonal entry b_{ii} in Y_n with a directed edge from the vertex i in G to the reference vertex r. Orientation of the edge is toward r and the value of the edge is:

$$\text{Edge value for diagonal entries} = \left(b_{ii} - \sum_{\substack{x = 1 \\ x \neq i}}^{n} b_{ix} \right).$$

c. Identify all off diagonal entries b_{ij} with an oriented edge directed from vertex i toward vertex j with value b_{ij}.

III. Find all directed trees of the graph terminating at the reference vertex. A directed tree is a tree with edges whose orientation is confluent toward the reference vertex.

IV. Compute the determinant Δ of Y_n as the sum of directed tree admittance products.

A very simple nonreciprocal network N shown in figure 4 will be used to demonstrate Chen's method.

The linear directed graph G corresponding to N is shown in figure 5.

The node admittance matrix Y_n for G written with vertex r as reference is:

$$Y_n = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} (y_1 + y_2) & 0 \\ y_g & y_3 \end{bmatrix} \end{array}.$$

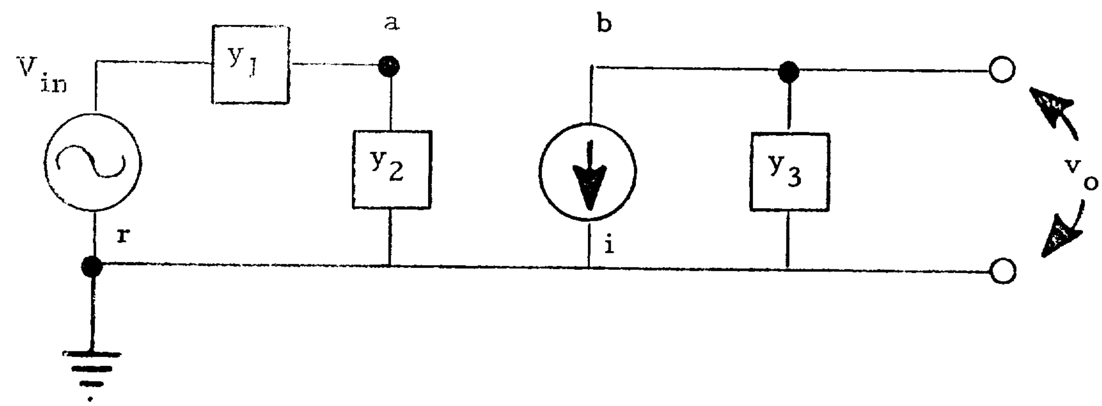


Figure 4.

Example Network N.

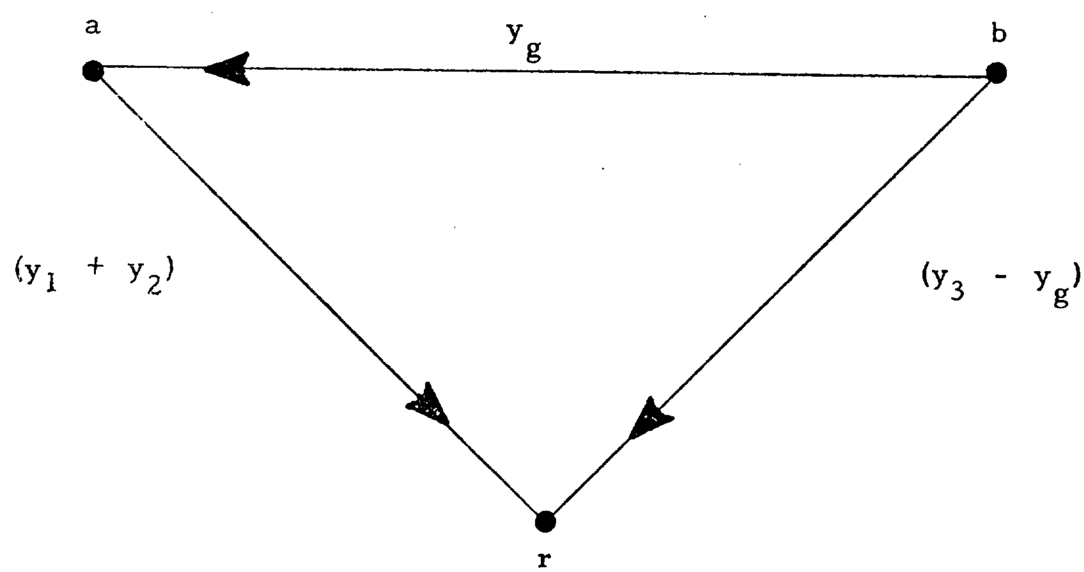
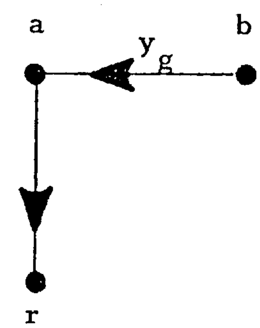


Figure 5.

Graph G of Example Network N.

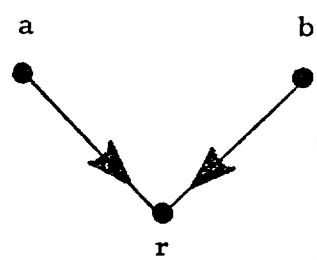
From graph G , it can be seen that some of the intuitive significance of the topological method is lost. The edges $(y_1 + y_2)$ and $(y_3 - y_g)$ are no longer simply identifiable with network parameters. The directed trees terminating on the reference vertex r are as follows:

$(y_1 + y_2)$



$$\Longrightarrow \quad y_g \, (y_1 + y_2) .$$

$(y_1 + y_2)$ $\quad$ $(y_3 - y_g)$



$$\Longrightarrow \quad (y_3 - y_g) \, (y_1 + y_2).$$

and therefore,

$$\begin{aligned} \Delta = \quad & y_g y_1 + y_g y_2 + y_1 y_3 - y_g y_1 - y_g y_2 + y_2 y_3 \\ & = (y_1 + y_2) \, (y_3). \end{aligned}$$

Here the interesting fact of cancellations pertinent to all topological methods for nonreciprocal networks is demonstrated. Although topological methods normally reduce cancellations and the attendant amount of computational labor involved in evaluating Δ, this is not always the case. No method of avoiding cancellations entirely has yet been devised.

Chen's method improves on the method of Mayeda [1] and Coates [4] by consolidating two graphs, the voltage and current graphs, into one directed graph. Chen's graph is nonetheless more complex than necessary because two edges are used for each bilateral element not touching the reference vertex. Our example did not show this fact because the reference vertex was common to all admittance elements.

Since Chen's stated objective was the reduction of the labor of computing the network determinant Δ and its cofactors, no attempt was made in his paper to discuss the formal relationships between the graph and the original network. Other investigators, however, have more than made up for this defficiency.

The importance of Chen's contribution lies mainly in the concept of the directed tree admittance products. Using this concept, he has avoided the difficult problem of determining the sign of the terms in the expansion of Δ.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF S. J. MASON.

Mason has presented a method of topological analysis of general linear networks which does not require any intermediate algebraic operations beyond those of network modeling. If one considers a simple network transformation, where unilateral elements are modeled by their mathematical counterparts, as topological, then the entire method is purely topological. This fact is significant to the extent that it allows a reduction in the amount of labor envolved in computing network functions.

When Mason developed his method in 1957, very little was being done in automatic analysis by computer and his work bears the flavor of this emphasizing its applicability to hand analysis. Since that time, a considerable interest has developed in computer aided analysis and many of the benefits of reduction in computational effort carry over to remain valuable in this new field but, since the computer is not as perceptive as the human, a considerable amount of the work requires revision before it can be considered compatible. Specifically the method requires simplification and the elimination of those steps which necessitate recognition of the more complex models of unilateral elements, the method can be divided into three parts: First; modeling of the original network with ideal elements and generating a linear partly oriented graph. Second; finding directed trees of the partly oriented graph. Third; computing network functions form the trees thus produced.

Procedure.

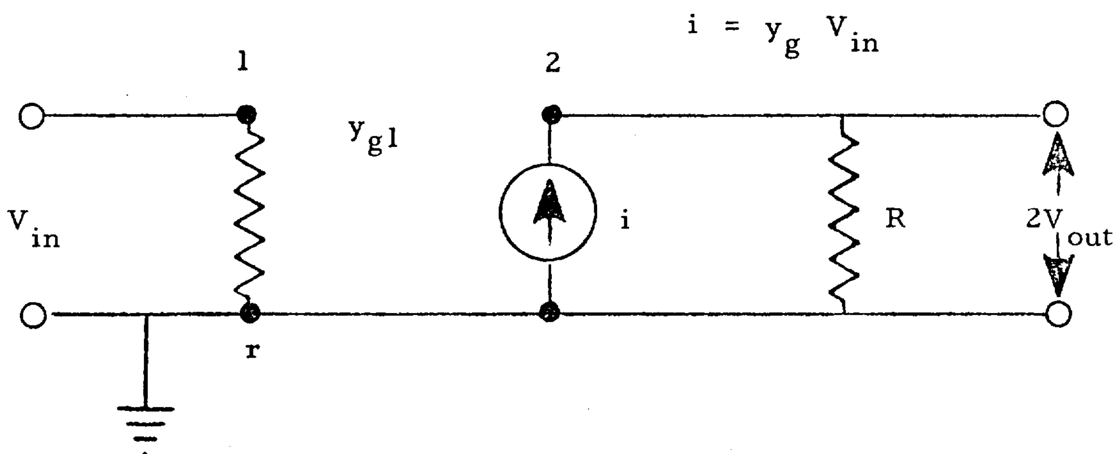
I. Start with a general network and reduce all unilateral elements to one of the following:

   a. A current source whose value depends on the voltage of one terminal (transadmittance).

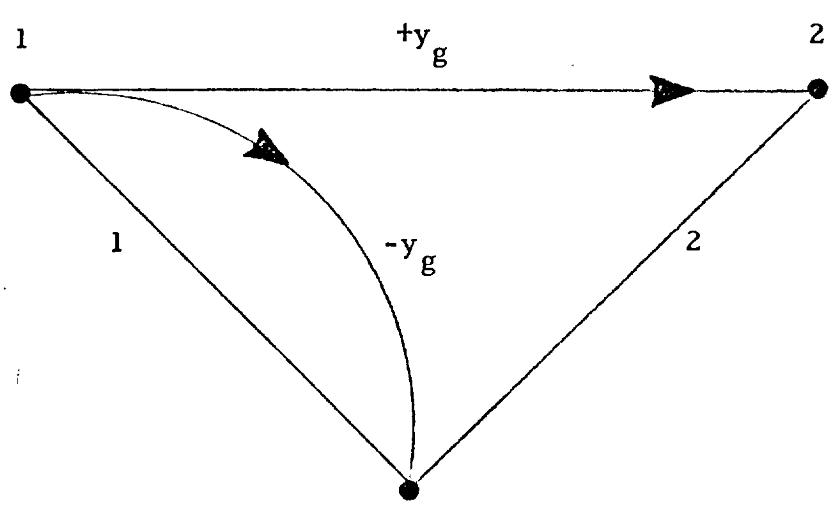
b. A current source whose value depends on the sum of the voltages found on each terminal (Gyristor).

c. Combinations of a and b above such as the Gyrator, (a confluent ring of 3 Gyristors) or the unator (a confluent ring of 3 transadmittances).

As an example of this process, consider a voltage controlled current source immeshed in a simple network.



The graph for this network using transadmittance modeling is:



Other more complex models could be used but this model has the merit of being the simplest because, in order to make an isomorphic transformation, it is necessary to account for all currents at vertex 1, but a transadmittance element from vertex 1 to vertex 2 alone does not accomplish this. It is necessary therefore in the general case to provide a transadmittance element from the control vertex to both the source vertex and the sink vertex of the generator.

II. Form a linear, partly oriented graph of the transformed network. In such a graph, bilateral elements are represented by undirected edges and unilateral elements by directed edges as indicated in I above.

III. For a reciprocal network, compute its node determinant as tree admittance products in the ordinary sense. Mason shows that the determinant can be expanded as the sum of all possible path cofactors for paths between any two vertices in the network. The path cofactors are tree admittance products for a new network where all vertices in the path are coalesced to a single vertex.

IV. For a nonreciprocal network, specify the reference vertex.

V. Compute all tree admittance products with the following modifications:

a. Multiply tree admittance products by $(-1)^m$ where m is the number of gyristors, if any, pointing away from the reference vertex in that tree.

b. Multiply a tree admittance product by zero if any unistor points away from the reference vertex in that tree.

Using the network of figure 1 as an example, a linear graph using the unistor or transadmittance element is formed as shown in figure 6 by the method of I above, and the trees of this graph are computed.

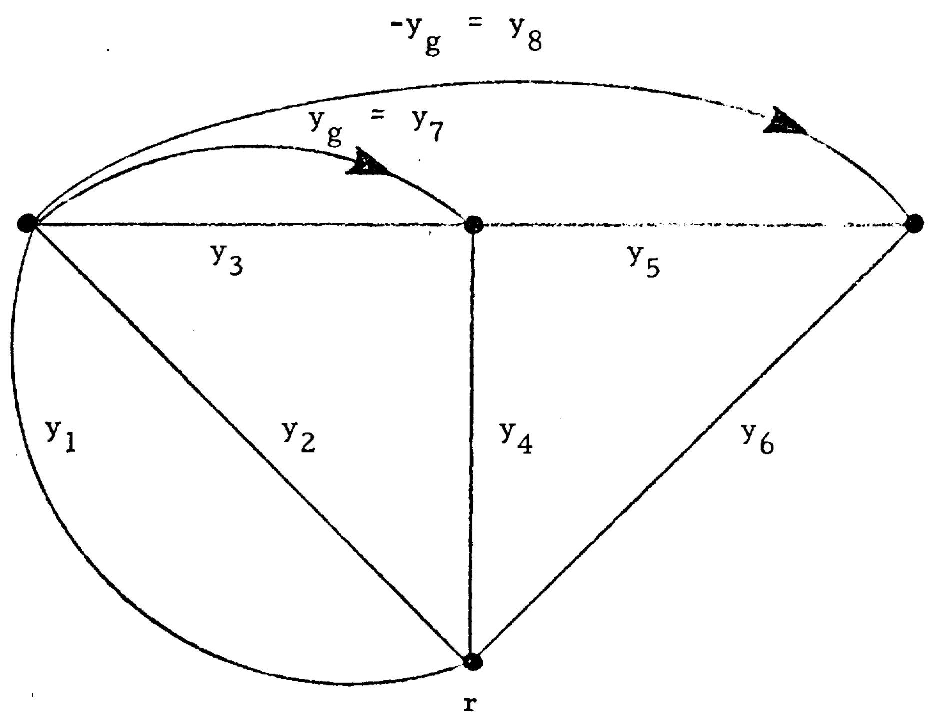


Figure 6.

Graph of the Network of Figure 1 Using Mason's Method.

Trees less those with directed edges pointing away from the reference vertex (such as 138, 148, 478, and 248) are:

135	235	345	458
136	236	346	467
145	245	356	468
146	246	368	567
156	256	457	368

After cancelling terms with equal magnitude and opposite signs such as $y_5 y_6 y_7$ and $y_5 y_6 y_8$ (so that $y_5 y_6 y_7 + y_5 y_6 y_8 = 0$), we have the determinant of the node admittance matrix:

$$
\begin{aligned}
\Delta = \quad & y_1 y_3 y_5 + y_1 y_3 y_6 + y_1 y_4 y_5 + y_1 y_4 y_6 \\
& + y_1 y_5 y_6 + y_2 y_3 y_5 + y_2 y_3 y_6 + y_2 y_4 y_5 \\
& + y_2 y_4 y_6 + y_3 y_4 y_5 + y_2 y_5 y_6 + y_3 y_4 y_6 \\
& + y_3 y_5 y_6 - y_g y_3 y_6 .
\end{aligned}
$$

Mason's method is a direct outgrowth of the concepts developed by Percival, Mayeda, Coates, and others, but it improves upon them by showing the simple relationships between the topological analysis of reciprocal networks and that of non-reciprocal networks. With some modifications and a restructuring of the basic orientation it forms the basis of an excellent method of computer aided analysis. Nathan has proposed an identical graph structure but a different and very much more complex method of finding "admissible" tree admittance products. It is not difficult to show that the partly directed graph used by both Mason and Nathan is the simplest graph representation possible for characterizing a network. It is clear that the basic topological quantity, the edge bounded by its two end points or vertices is the simplest model for bilateral elements. The directed edge is likewise the simplest model for unilateral elements. Since most generators require two directed edges, one might question whether it is possible to model such a generator with only one edge. It is indeed possible but, the dependence relationships which are established would have to be conveyed by means other than topological, such as a 2 x 2 matrix or simultaneous equations. Since this is non-topological information, it is eliminated as a possibility. The two directed edges are thus seen to be the simplest purely topological model which expresses all dependence relationships.

One of the primary objectives in Mason's paper is to show the means for graphically expanding the terms of the determinant of the node admittance matrix. This fact has subdued the more important general topological method emphasized here and has led to a misunderstanding on the part of some that it is not directly related to the mainstream of developments in topology. Seshu and Reed [10] have dismissed the method unfairly in a summary manner and emphasized the compartively cumbersome method of Mayeda et. al. Though the topological expansion of determinants may have great future significance, especially in computer manipulation of matrices, the importance of the direct topological analysis by means of trees of a partly oriented graph should not be underestimated.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF A. NATHAN.

Nathan's method is similar to that of Mason with the exception of the manner in which tree admittance products are obtained. Nathan requires use of newly defined topological entities such as "loop-trees" and "loop-woods" which challenge the imagination with the very unseemly juxtaposition within a single term of the concepts of directed loops and directed trees. Although the rules of formation of looptrees and loopwoods are straightforward enough, they are nonetheless complex to the point that diminishes the plausibility for their use in computer programs. Recognition of the loopwoods without error is very difficult for a hand analysis of a network of intermediate size. Any systematic procedure based on this method would be required to identify all directed loops as well as directed trees. In addition it would also be required to identify sets of nth order loops in a manner similar to Mason's gain formula. This is a computational hazzard. The benefit of doing so, however, lies in the fact that many fewer trees need be found where directed loops exist. This corresponds roughly to removing common factors in the expansion of the determinent of the network. Since finding loops is as difficult as finding trees by computer, the method is not ideally suited to machine implemintation. It would be better to find only loops or only trees.

In his paper, Nathan makes the claim that a desirable objective which he seeks is the determination of network quantities from a graph which is topologically identical to the given network and indeed he has come remarkably close to accomplishing this task, but some differences do and must exist in certain cases. Because he models the unilateral elements with transadmittances, the network transformation listed in the following procedure is required. Nathan's recent paper [16] seems to have ignored the fact that Mason's [3] has proposed the same partly directed graph. Nathan says, referring to signal flow graphs, ". . . this graph is much more complicated than that of the original network since it replaces each reciprovcal branch by two parallel branches". All this is true for signal flow graphs, but in reference [3] Mason very briefly has shown that the simplest graph of a general network can be used to compute the determinant of the node admittance matrix. In this Mason has shown the way and Nathan has subsequently and correctly concurred. Nathan has succeeded in reducing difficulty in sign determination. All sign information is taken into account in the formation of the loopwoods.

Procedure.

I. Convert all unilateral network elements into equivalent transadmittances as shown earlier for Mason's method.

II. Draw a linear partly oriented graph of the transformed network.

III. Find all directed loops - these are loops of directed edges each of which is confluent with the others.

IV. Find all directed loop trees - these are directed trees with reference vertices. Compute loopwoods of all orders. These are sets of loop trees common to loops found in III above. The loopwood of order zero degenerates to the set of ordinary directed trees of the network. The first order loopwood corresponds to the 2-trees of ordinary topological analysis with the exception that one loop at a time containing a gain or unilateral element is included. Higher order loops taken two at a time, three at a time, etc. are treated in a manner similar to that used in Mason's gain formula for signal flow graphs.

The rules of determining the symetric and asymetric cofactors of the node determinant by this method are quite complex. For those who desire a more detailed description, reference [6] serves as the best guide.

Let us consider again the example of figure 1. Our first step is to construct a suitable mathematical model where the generator is represented by two transadmittance elements; one supplying positive current to the network and acting like a source and the other supplying negative current and acting like a sink. Care must be taken here not to confuse the direction of current flow with the sense of direction of the dependence relationships of the elements y_g and $-y_g$. Each originates at the control vertex to preserve these relationships necessitated by the network structure. All this is shown in figure 7.

The same linear graph as shown for Mason's technique in figure 6 applies to this model:

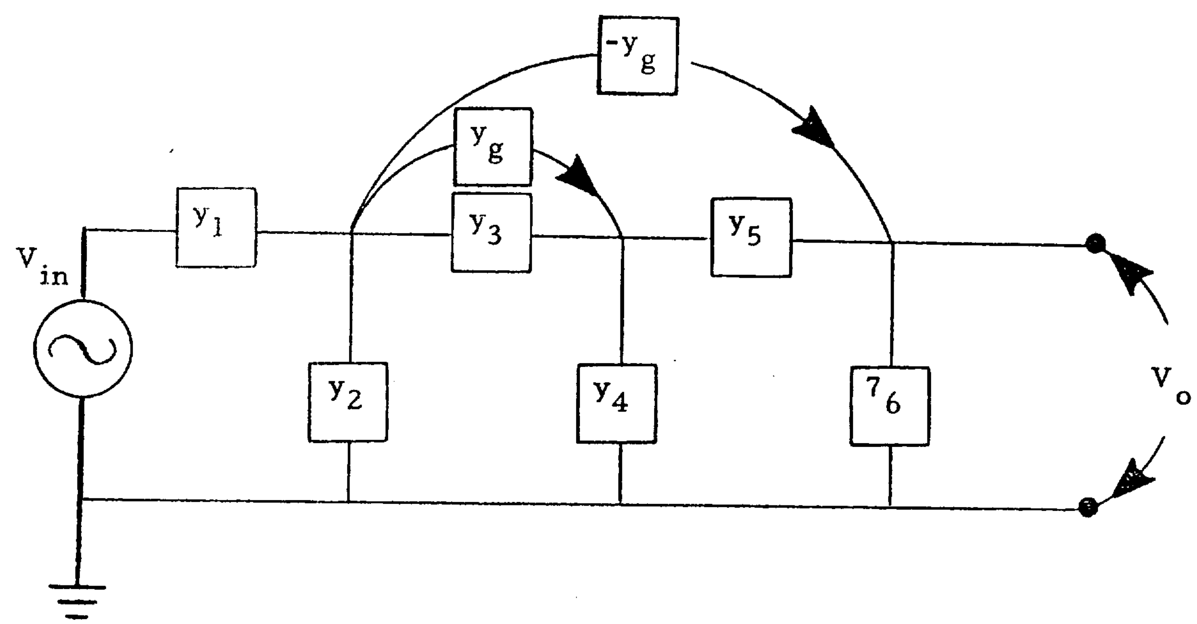


Figure 7.

Transformed Version of the Example Network.

The next step in evaluating the determinant of the node admittance matrix is to consider the partial graph with all directed edges removed. This is shown in figure 8 and the trees of this graph produce some of the terms in the expansion of the determinant corresponding to the loopwoods of order zero expressed in Nathan's terminology as:

$$L_{(k)}^{(o)} = \sum \left(\begin{array}{l} \text{Tree admittance} \\ \text{products of modified} \\ \text{graph without direc-} \\ \text{ted edges} \end{array} \right).$$

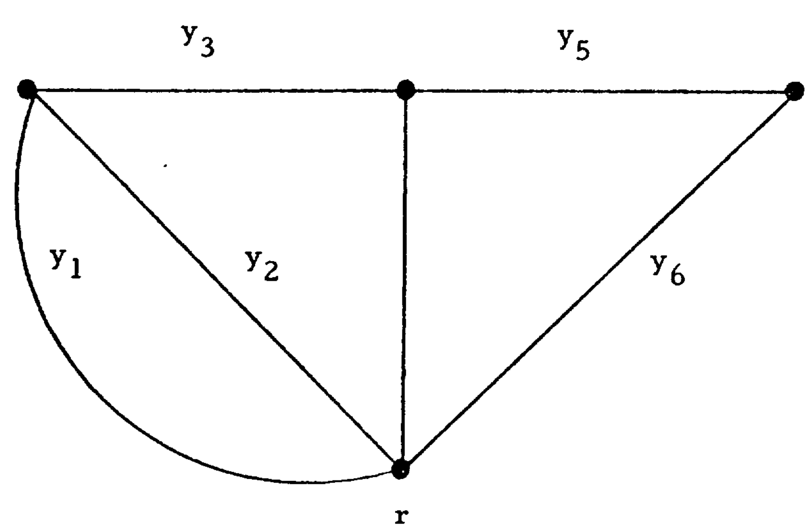


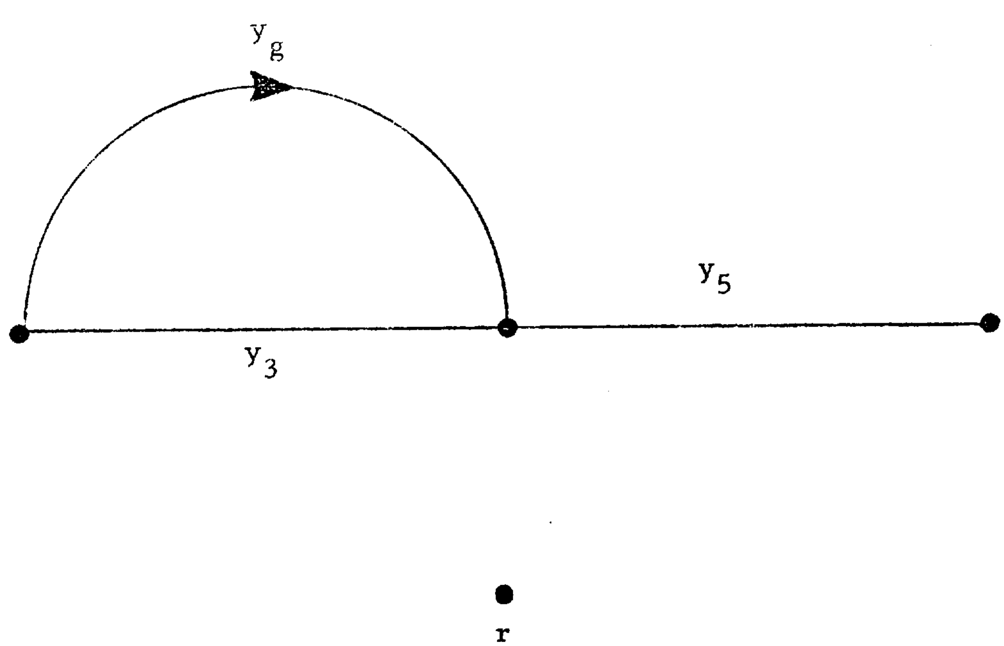
Figure 8.

Partial Graph of Figure 7.

The trees are:

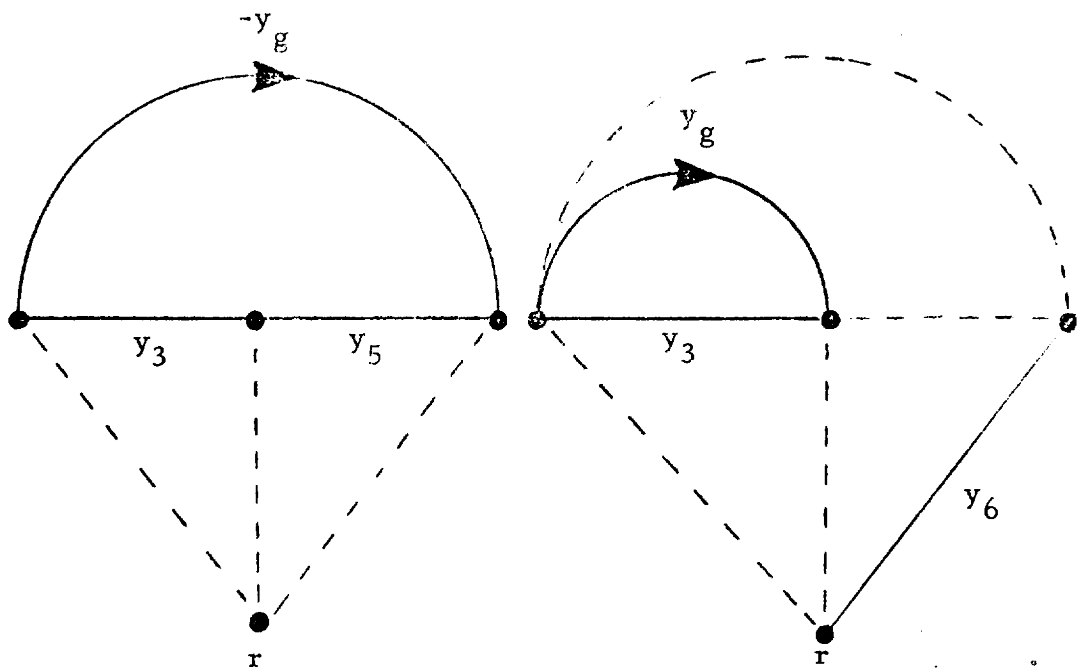
135	156	245	345
136	235	246	346
146	236	256	356.

The next and somewhat more complex step is to find all loops taken one at a time and evaluate the loop trees or directed trees which terminate on these loops. One such directed loop tree having the reference vertex as its reference is:



If it were not for the allowable loop, this would be a 2-tree, with the reference vertex in one part and all other vertices in the other part.

Similarly, the other first order loop trees are found as shown on the next page.



Taken together these are the first order loopwoods and the terms produced are:

$$\text{Loopwood of the 1st order} = -\sum_k L^{(1)}_{(k)} = y_g y_3 y_5 - y_g y_3 y_5 - y_g y_3 y_6$$

$$= - y_g y_3 y_6 .$$

And, since there are no second order loopwoods, the procedure terminates therefore;

$$\begin{aligned}\Delta = \; & y_1 y_3 y_5 + y_1 y_3 y_6 + y_1 y_4 y_5 + y_1 y_4 y_6 \\ & + y_1 y_5 y_6 + y_2 y_3 y_5 + y_2 y_3 y_6 + y_2 y_4 y_5 \\ & + y_2 y_4 y_6 + y_2 y_5 y_6 + y_3 y_4 y_5 + y_3 y_4 y_6 \\ & + y_3 y_5 y_6 - y_g y_3 y_6 .\end{aligned}$$

It requires a great deal of care to perform these computations by hand but, after much practice the method could be found helpful. It does have the advantage of producing factored form results but for computer analysis, it seems unduly weighted with diverse procedures and sub-procedures. A more direct method is much to be preferred.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF A. TALBOT.

Talbot has proposed a mixed method of analysis, partly algebraic and partly topological. The algebraic portion starts with the development of the edge admittance or impedance matrix which, in general, is not diagonal. This matrix is rearranged in such a way that all the necessary relationships between coupled elements of non-reciprocal sets are expressed as sub-determinants of low order usually 2 x 2 falling on the main diagonal. From the sub-determinants, selection criteria for determining the admissibility of terms computed from the trees of the network is derived. The topological aspects of the method depend essentially on evaluating trees of the network and signs of the terms represented by admissible trees, from the point of view of computer aided analysis, the method is cumbersome because many different procedures, including those for manipulating matrices, are required and the essential benefits of simplicity of the topological technique have been sacrificed.

Procedure.

I. Transform all unilateral or active elements to square matrix admittance parameter representation.

II. Form the network branch matrix in such a way that all passive elements are single uncoupled elements entered on the main diagonal and active or unilateral elements consisting of square matrices are entered on the main diagonal. (It may be required to change edge order to accomplish this.) The resultant matrix should have the form shown on the next page.

$$\begin{bmatrix} [\,y_1\,] & & & & & O \\ & [\,y_2\,] & & & & \\ & & \cdots & [\,y_i\,] & & \\ & & & & \cdots & \\ O & & & & & \end{bmatrix}$$

where some of the $\left[Y_k\right]$ (where k = 1, 2, . . .) may be matrices of low order and others may be single entries representing individual elements.

III. Form a selection table from the branch matrix showing all possible combinations of parameters Y_1 , Y_2, . . . and compute the determinants pertinent to each of these combinations. These determinants will form admittance products for the elements or coupled networks.

IV. Draw a linear graph of the network from the entries in the branch matrix.

V. Compute all trees of the graph.

VI. Form a selection table listing all possible combinations of trees taken 2 at a time and corresponding to entries in rows and columns respectively of the branch matrix. Each combination realized produces a set of elements identifying a tree admittance product. Admittance parameters from III can be back substituted at this point.

VII. Compute sign term for each admittance product according to Section V p. 175-179 of Talbot's paper [7].

VIII. Compute determinant of the network as:

$$\sum_{\text{all trees}} \left(\text{Tree admittance products} \right).$$

IX. For cofactors of the node determinant, a new edge matrix must be formed and a new graph corresponding to this matrix must also be formed. The process of computing follows steps II through VIII.

Because this method is not entirely topological and relatively unsuited to computer analysis, no example will be given here but, the interested reader is referred to Talbot's paper [7] where a number of good examples are given.

TOPOLOGICAL ANALYSIS OF GENERAL NETWORKS BY THE METHOD OF M. T. JONG AND G. W. ZOBRIST.

Jong and Zobrist have proposed an extension to the methods of Mayeda and Coates with some basic differences in approach. The main differences are that only one network graph is required, not two, and higher order trees must be computed where the networks contain active or non-reciprocal elements. Using just one graph is a definite advantage but finding the higher order trees in a systematic and efficient manner is a problem yet to be solved. At the time of this writing, Jong has developed the method fairly well for non-reciprocal elements in the form of transadmittances, but the extension to networks containing more than one dependent voltage source has not yet been done. This is no fundamental limitation but it impedes the process of evaluation.

Considering networks with only transadmittance elements and bilateral elements, the computation of the determinant of the node admittance matrix by the method of Jong and Zobrist starts by forming all the tree admittance products of the passive sub-graph. Then higher order trees are obtained to express those relationships caused by the transadmittances. The following procedure will be illustrated with the aid of the example network of figure 9.

Procedure.

I. To find the determinant of the node admittance matrix, first separate the network into active and passive sub-networks N_a and N_p as shown in figure 10.

II. Generate the linear graph G_p of the passive network N_p as shown in figure 11.

III. Compute the trees of the passive sub-network.

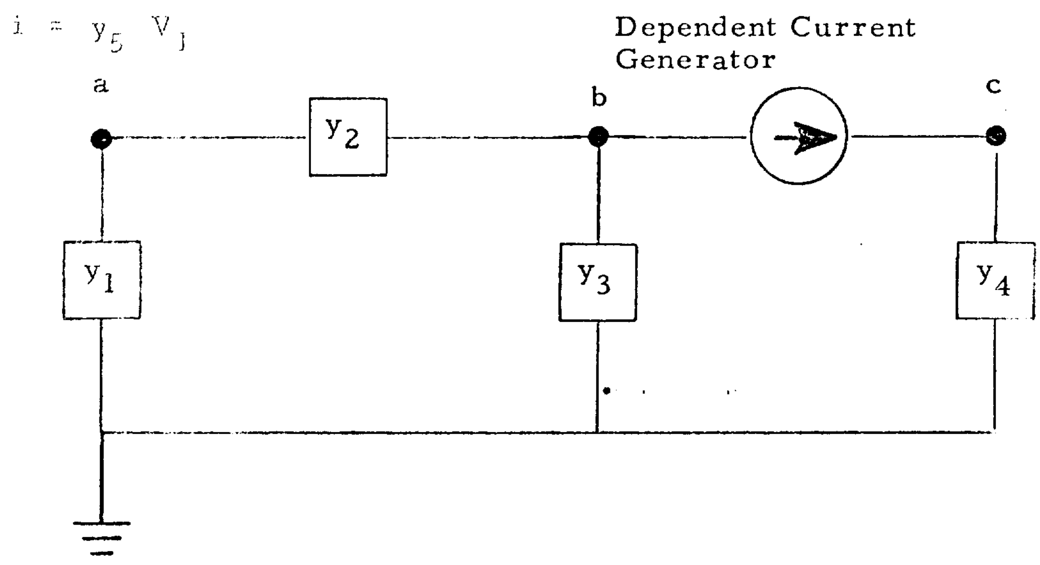
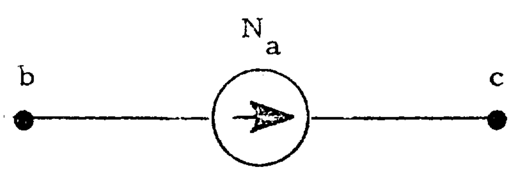
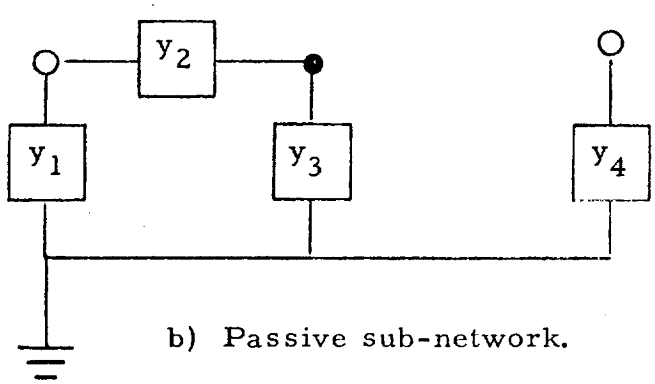


Figure 9.

Example Network.



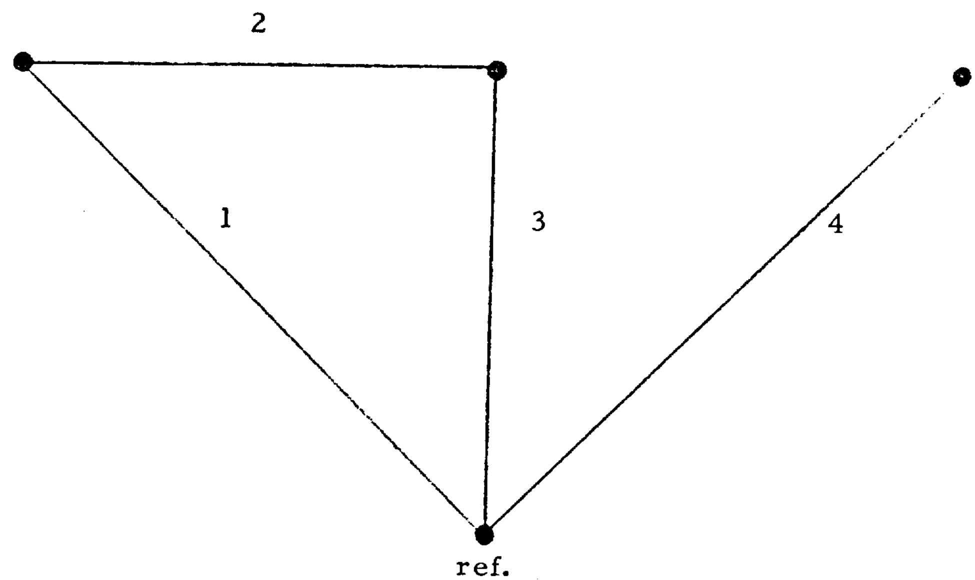
a) Active sub-network.



b) Passive sub-network.

Figure 10.

Active and Passive Sub-Networks for the Example.



G_p

Figure 11.

Graph of the Passive Sub-Network.

For our example, the trees of N_p are found by permuting edge numbers as follows:

-1-	-2-	-3-	Ckt
1	2	4	
1	3	4	
2	3	4	

From these, the first set of terms in the expansion of Δ is produced:

$$T^o \qquad = \qquad y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4$$

IV. We now proceed to find the next higher order k-trees which in this case is the set of 2-trees where vertices are separated in a special way as illustrated here for k = 1.

To find T^1 (the set of 2 trees required) it is necessary to separate the graph G_p into controlling vertices and controlled vertices of the transadmittance elements:

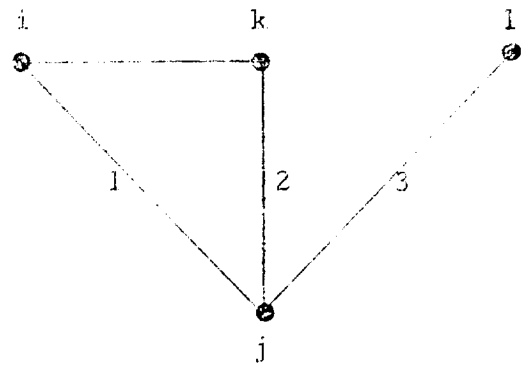
i = + controlling vertex
j = - controlling vertex
k = + controlled vertex
l = - controlled vertex

Then, we find sets of 2-trees with controlling vertices separated and also with controlled vertices separated and take the intersection of these two sets to produce the required higher order set. VIZ:

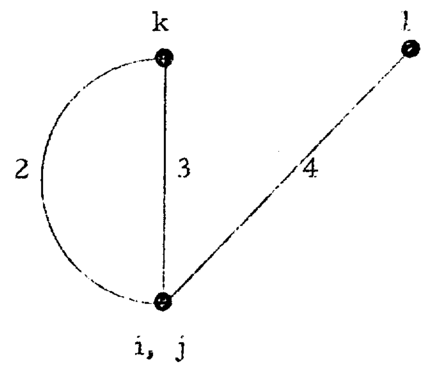
$$T^1 = T_{2_{i,\,j}} \cap T_{2_{k,\,l}}.$$

$T_{2_{i,\,j}}$ is found by identifying the vertices of the graph G_p and computing as shown in figure 12.

The 2 trees $T_{2_{i,\,j}}$ and $T_{2_{k,\,l}}$ are found from the modified graphs shown on the following pages.

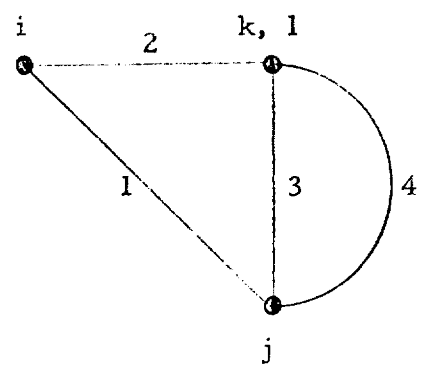


(a)



$$\Longrightarrow \quad y_2\, y_4 + y_3\, y_4 = T_{2_{i,\, j}}$$

(b)



$$\Longrightarrow \quad y_1\, y_2 + y_1\, y_3 + y_1\, y_4 + y_2\, y_3 + y_2\, y_4 = T_{2_{k,\, l}}$$

(c)

Figure 12.

(a) Controlling Vertices ij and Controlled Vertices kl of the Graph G.

(b) 2-Tree Products Produced by Identifying i with j.

(c) 2-Trees Products Produced by Identifying k with l.

and,

$$T^{I} = \left[y_2 y_4 + y_3 y_4 \right] \cap \left[y_1 y_2 + y_1 y_3 + y_1 y_4 + y_2 y_3 + y_2 y_4 \right]$$
$$= y_2 y_4 .$$

Note: In this step, it was necessary to eliminate five unnecessary 2-trees - a wasteful effort.

V. If there were more transadmittance elements, higher order sets of k-trees would be required. However, for the example k = 1 and the 2-trees terminate the search. Vertex separation for higher order trees is more complex and the interested reader is referred to the paper of Jong and Zobrist for clarification on this point.

VI. Δ is found by summing all tree admittance products thus produced according to the formulation following on the next page:

$$\Delta = T^{o} + \sum_{r=1}^{r} y_{m^r} T^{r}$$

$$+ \sum y_{m1} y_{m2} T^{12} + y_{m1} y_{m2} y_{m3} T^{123}$$

$$+ \ldots + y_{m1} y_{m2} \cdot \qquad y_{mr} T^{12.v}$$

where:

Y_{mr} = The rth transadmittance element.

T^{o} = Sum of all tree admittance products of the graph G_p.

T^{r} = Sum of all 2-tree admittance products separating the network N_p into two parts. On containing the + and - input vertices. The other containing the + and - output vertices of the rth transadmittance element.

$T^{r1, r2}$ = Higher order tree admittance products as explained in text.

For our example, this is simply:

$$\Delta = T^{o} + T^{1}$$

$$\Delta = y_1 y_2 y_4 + y_1 y_3 y_4 + y_2 y_3 y_4 + y_2 y_4 y_5.$$

The complexity of computing higher order k-trees increases very rapidly as the number of unilateral elements is increased. This is shown dramatically by computing the number of sets of higher order trees as a function of k as shown in table 1.

It is clear that for an active network with four or more generators, the cost of computing k + 1 tree admittance products considering that all N (k) sets required would be prohibitive. Many of these combinations may produce zero terms. This is evidence of the fact that some efficiencies in the computing process are possible. The method in general has promise but also some difficulties, the greatest of which seems to be the difficulty of finding the higher order k-trees where vertices are separated into many different required arrangements. When this problem is solved, it may be possible to make good use of the technique for computer application but at the present time and for this reason, it is not suitable.

Table 1.

Number of Sets of k-Trees as a Function of k.

k	N (k)
1	1
2	4
3	24
4	192
5	1920
6	23040

COMPARISON OF THE METHODS OF GENERAL TOPOLOGICAL ANALYSIS DISCUSSED ABOVE.

Each of the methods of topological analysis can be divided into three broad task areas involving first, formation of a mathematical model; second, generation of a linear graph, and third, determination of admittance parameters. Each differs from the others in some if not all of these categories. Model formation starts with the initial network to be solved and consists of reducing the network to a form wherein each element can be represented by an edge of the corresponding linear graph. The linear graph is then derived and the required network functions are determined from the graph by some systematic method. On the basis of these categories, a comparison between methods will be attempted.

It would, of course, be desirable to avoid the necessity for transforming a network into a form suitable for generating a linear graph but unfortunately no one has as yet come up with a satisfactory method to do this; furthermore it is conceptually not possible to relate the required information from a dependent source to linear graph form without adding an element not usually found in the schematic of a network, namely that element required to show the dependency relationship. The easiest method of establishing this dependency relationship seems to be to use the transadmittance element or voltage controlled current generator with the controlling voltage existing at one of the terminals of the current generator. Mason has adopted this approach but has also included a redundant method using the Gyristor. Since any unilateral element can be formed using the transadmittance element alone, and since transadmittance has a simple physical significance, it seems unnecessary to include the Gyristor as a special case. The method of using small matricies to describe unilateral elements as done by Talbot and Brown is comparitively cumbersome.

The dual voltage and current graphs of Mayeda and of Coates contain a great amount of redundancy. Here each bilateral element is represented by two edges. Even though Chen's method uses a single graph, his graph still contains two edges for each bilateral element. By far the simplest graph structure is that proposed by both Mason and Nathan using undirected single edges for passive elements. Although the graph structure proposed by Brown is relatively simple, each edge is directed and it is necessary to keep account of direction in determining admittance parameters. In the next section, we will develop a method using partly oriented graphs similar to those of Mason and Nathan.

All topological methods reviewed in this paper make use of the concept of a "tree" of some kind. Starting with the methods of Mayeda and Coates, the concept of common tree was introduced. This concept was greatly simplified by the directed tree of Chen. Nathan defines loop-trees and complicates the picture unnecessarily. Since all of these methods use "trees" or loop trees in determining admittance parameters, it would seem that the simplest tree-finding algorithm should be the most effective. In addition to trees, Talbot and Brown find it necessary to include non-topological data formed from the matrix equivalents of unilateral elements in the computation of admittance parameters and consequently produce answers at the expense of considerable additional computational labor. Jong and Zobrist have suggested the use of sets of k + 1 trees, finding such tree sets is as yet a partly unsolved problem.

Inherent in the computation of admittance parameters is the problem of determining sign. The complexity of sign determination seems to be directly related to the complexity of the tree structure recognized by each method. Chen has provided methods where the sign determination is made directly from the tree admittance products. Brown claims that no sign problem exists, but if one investigates the accounting of edge orientation required in determining the admittance parameters for coupled elements, one can recognize all the symptoms. Talbot's sign problem is not as complex but still requires the comparison of orientation of tree pairs.

Taken altogether, it would seem that the best features of all methods discussed above should be synthesized into a simple topological technique that requires the least amount of effort to use and affords the greatest insight into the analysis problem. This method would use simplest possible transadmittance formation concepts, the partly oriented graph structure proposed by Nathan and Mason, and the simplest possible admittance parameter determination methods of ordianary passive network analysis. Such a method is proposed in the next section.

DEVELOPMENT OF A NEW METHOD FOR TOPOLOGICAL ANALYSIS OF LINEAR ACTIVE NETWORKS.

Nathan [6] and Mason [3] have proposed methods of topological analysis which have some similar recommending features. Each uses a partly oriented graph and each avoids the difficulty of computing the sign terms in the expansion of determinants. The method outlined here is related to each of the methods discussed earlier but owes more to Mason and Nathan than to the others. It will be shown that this method is a specialization of the method of Mayeda [1] retaining all the generality required to analyze any linear active network. This is possible because, however elegant Mayeda's method may be, it contains redundant graph elements and redundant sign terms. The procedure for this method is very simple starting with network modeling, continuing with generation of a graph, and concluding with the topological determination of network determinants and cofactors as shown below for nodal analysis.

Procedure.

Step 1.

Model all dependent elements by using one or a combination of transadmittance elements (a transadmittance element is a voltage controlled current source with the independent "+" voltage terminal attached to the current source and the "-" voltage terminal attached to the reference or ground terminal). If a dependent voltage generator is encountered it will be necessary to obtain the norton equivalent before transformation to transadmittance form. Any transadmittance directed from a vertex to the reference vertex must be replaced by a bilateral edge of the same value. Justification for this operation will be given later.

Step 2.

Form a partly oriented graph of the transformed network using undirected edges to represent bilateral elements and directed edges to represent unilateral elements.

Step 3.

To compute Δ, the determinant of the network, find all directed trees of the partly oriented graph of the network and from these, compute the sum of the directed tree admittance products.

Step 4.

To compute cofactors of the node admittance matrix, find all directed 2-trees of the partly oriented graph of the network and from these, compute the sum of the directed 2-tree admittance products.

A dual procedure is possible for loop analysis using the transimpedance unilateral element model to compute the determinant of the loop impedance matrix and its cofactors. However, we will illustrate the method only for nodal analysis because it is relatively easy to transform from one to the other.

Mathematical Development.

Before we can build the mathematical foundation to support this new method of topological analysis of general linear networks, it is necessary to define certain terms, most of which have already been defined elsewhere but are repeated here for clarity. Some new terminology, however, is needed to express several new concepts necessary in the development.

Definitions.

1.	edge	A line segment with two endpoints used to topologically represent a mathematical or physical relationship.
1 a.	undirected edge	Any edge which graphically represents a bilateral mathematical or physical quantity. In such an edge, no distinction is made between either of the two endpoints.
1 b.	directed edge	Any edge which graphically represents a unilateral mathematical or physical quantity. A directed line segment is used to represent directed edge and a distinction is made between the vertex of departure and the vertex of arrival of such an edge.
1 c.	controlling edge	An edge of a control graph, graphically representing the voltage control function for a transadmittance.
2.	vertex	An endpoint of an edge.
2 a.	vertex of departure	The vertex nearest the tail end of a directed edge.
2 b.	vertex of arrival	The vertex nearest the head end of a directed edge.

2 c.	<u>reference vertex</u>	A selected vertex within a graph, which has been selected as datum.
2 d.	<u>extreme vertex</u>	Any vertex within a graph, which has one and only one edge attached to it. A vertex of degree 1.
3.	<u>transadmittance</u>	A mathematical model of a voltage controlled current generator represented topologically by directed edge wherein current is thought to flow in an amount proportional to the voltage that exists at the vertex of departure of that edge.
4.	<u>graph or linear graph</u>	An ensemble of edges which, if connected, are connected at vertices only.
4 a.	<u>undirected graph</u>	A graph wherein all edges are undirected
4 b.	<u>directed graph</u>	A graph wherein all edges are directed.
4 c.	<u>partly directed graph</u>	A graph wherein the edges are either directed or undirected or combinations of both.
5.	<u>subgraph</u>	A graph which lacks one or more edges of the graph to which it is related.
6.	<u>control graph</u>	For every partly directed graph, one control graph exists to graphically model the control relationships of transadmittances within the original graph.

This control graph is formed by first replacing every undirected edge with two oppositely directed edges between the same terminals. Then, the head end of each directed edge within the graph is disconnected and subsequently reconnected to a chosen reference vertex. The control graph is therefore a directed star graph where all edges are directed toward the reference vertex.

7. directed path — A sequence of edges which are connected in such a manner that all edges are directed in the same way and that all the vertices involved are of degree two except the initial and the terminal vertices which are of degree one.

8. tree — A circuitless connected, subgraph which contains all the vertices of a graph.

9. directed tree — A tree of a partly directed graph wherein all paths between extreme vertices and a reference vertex are directed paths terminating on the reference vertex.

Transadmittance Model.

It is theoretically possible to model any real electrical network using the transadmittance model alone although for many reasons it may not be desirable to do so. We will, however, make this assumption that for nodal analysis, we can reduce any unilateral or bilateral element to an equivalent transadmittance model. Figure 13 shows the ordinary network model of a voltage controlled current generator along with the associated transadmittance model. Here it can be seen that two oriented edges are required to show the source and sink nodes of the current generator with their relationship to the control vertex a, the point

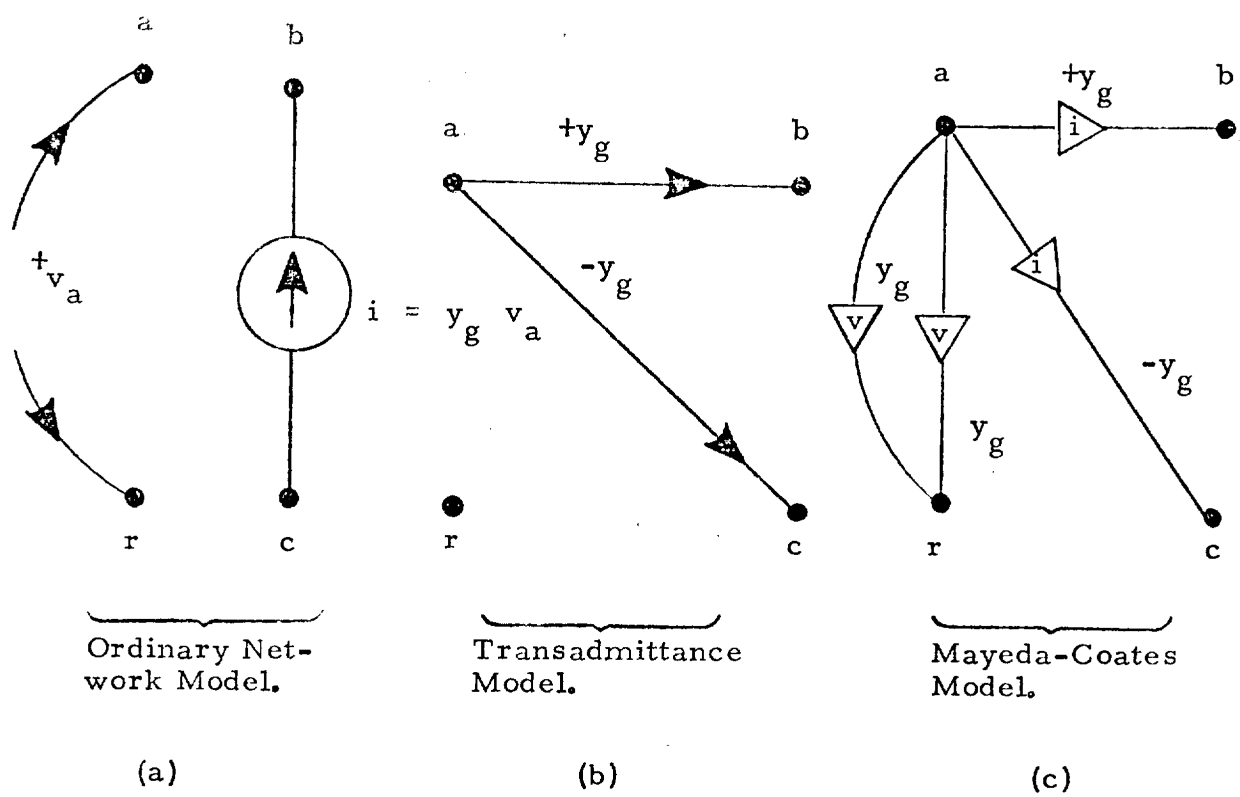


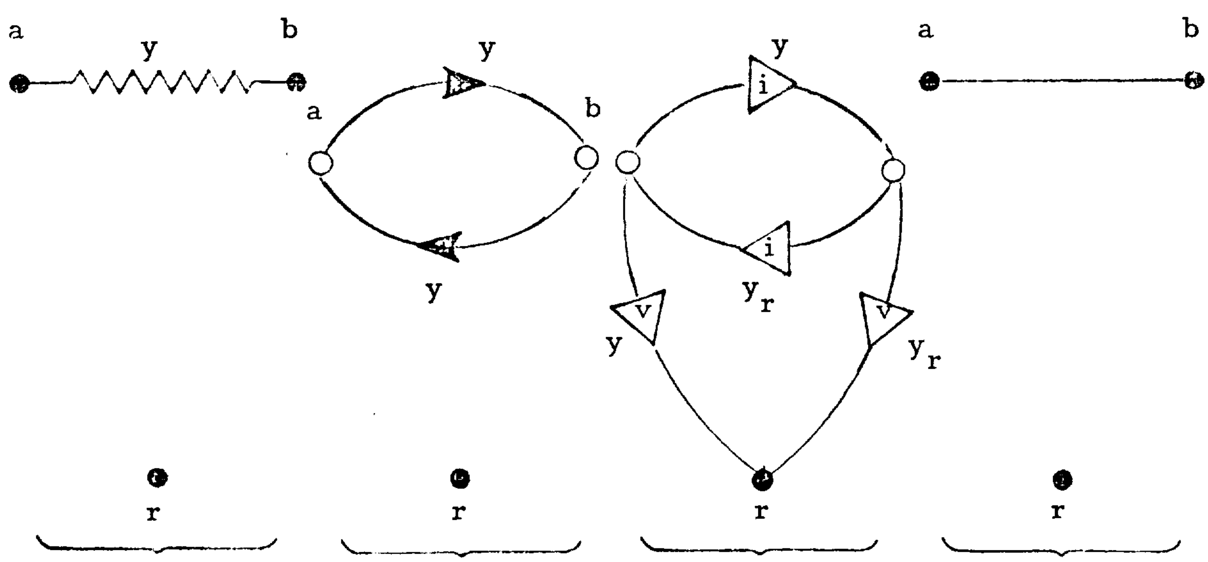
Figure 13.
Model of a Transadmittance Element

at which the control voltage v_a is applied. Also shown is the earlier and somewhat redundant Mayeda-Coates model indicating the current and voltage graphs superimposed.

A bilateral element such as a resistor or capacitor can be modeled as shown in figure 14 using transadmittance alone. However it is not necessary because of symmetry to use two directed edges. Only one undirected edge is required to express all the necessary and sufficient information required for analysis purposes.

Transformers and many other practical networks can be modeled by combinations of unilateral and bilateral elements but the details will not be taken up here.

Using the transadmittance model, we can represent practical networks with a combination of oriented and non-oriented edges such that



Ordinary Network Model. (a)

Transadmittance Model using directed Edges. (b)

Mayeda-Coates Model. (c)

Undirected Edge Model. (d)

Figure 14.

Models of a Bilateral Element.

the essential and none of the nonessential information is conveyed. We shall now show how a partly directed graph of a network can be formed and used to solve for the determinant of the node admittance matrix and its cofactors but first, we shall formally demonstrate that the partly directed graph model is justified.

Establishing a Relationship Between a Directed Graph and a Partly Directed Graph.

Let us start by considering the following kind of network with its descriptors.

Given:

1. A network N containing unilateral and bilateral elements.

2. A transadmittance model M of the network.

3. A directed graph G_d of the transadmittance model M. (All edges of G_d are directed.)

4. A partly directed graph G of the model M where all passive elements are represented by undirected edges and all active elements by directed edges.

Theorem 1: G_d and G are topological equivalent in the following way. All incidence relationships necessary and sufficient to describe the network are preserved in a transformation from G_d to G. Also, the rank and nullity of G_d is equal to the rank and nullity of G.

Proof: The graph of a transadmittance model of a bilateral element using oriented edges is redundant. This is so because the two directed edges of the graph (refer to figure 14) are identical except for direction which can be interchanged without loss of generality. Representing this graph by one undirected edge simplifies the structure and preserves the necessary incidence relationships.

The incidence matrix A representing G contains one column for each passive bilateral element and the incidence matrix A_d representing G_d contains two columns, identical except for sign, for each passive bilateral element. A and A_d are otherwise identical and it follows that the rank of each is the same because by elementary operations performed on the columns of A_d, A can be formed. It also follows that the nullity is the same because all vertices connected by directed edges in A_d are connected by nonoriented edges in A.

Computing Δ from the Partly Directed Graph.

In order to prove that we can compute the determinant of the node admittance matrix from the directed trees of this partly directed graph, we will first recall a relationship developed by Mayeda and Coates as follows:

$$Y_n = A_i \; Y \; A_v^t \tag{1}$$

where

A_i = the incidence matrix of the current graph which, in the case of an oriented graph of an exclusively transadmittance model, is the incidence matrix of the graph itself.

and

Y = the diagonal edge transadmittance matrix where unilateral as well as bilateral elements appear only on the main diagonal.

and

A_v = the incidence matrix of the voltage graph which in the case of an exclusively transadmittance model is the incidence matrix of the control graph.

Consider the relationship between transadmittance models of a network together with the Mayeda-Coates models containing voltage and current generators as shown in figures 13 and 14. There is a 1 to 1 correspondence between the elements of the transadmittance models and the current generators of the Mayeda-Coates models. This allows us to equate A; the incidence matrix of the current graph in equation (1) with A the incidence matrix of the directed graph of the transadmittance model. Transforming the transadmittance model into its control graph in the manner indicated in definition 6, page we produce a graph which is identical to the voltage graph of the Mayeda-Coates Model, therefore, $A_{c,\ r}$ the incidence matrix of the control graph of a transadmittance model is identical to A_v the incidence matrix of the voltage graph of the Mayeda-Coates model and we can specialize equation (1) to the case of the directed graph according to equation (2). This result is also applicable to the partly directed graph according to theorem 1.

$$Y_n = A\ Y_e\ A^t_{c,\ r} \tag{2}$$

where

A = a_{ij} = incidence matrix of the partly directed graph.

where

$A = a_{ij}$ = incidence matrix of the partly directed graph.

where

$$a_{ij} = \begin{cases} -1 \text{ for each undirected edge } j \text{ contacting the ith vertex or each directed edge } j \text{ departing from the ith vertex.} \\ +1 \text{ for each directed edge } j \text{ directed toward the ith vertex.} \\ 0 \text{ otherwise.} \end{cases}$$

Y_e = diagonal edge admittance matrix which may contain any number of transadmittance elements.

$A_{c,\ r}$ = the incidence matrix of the control graph of G with reference vertex r. The justification for this specialization is according to theorem 1.

By the Binnet-Cauchy theorem, we know that Det Y_n is the sum of the products of corresponding majors of A Y_e and $A_{c,\ r}$.

$$\text{Det } Y_n = \sum \left(\begin{array}{c} \text{Products of corresponding} \\ \text{majors of } A\ Y_e \text{ and } A^t_{c,\ r} \end{array} \right) . \qquad (3)$$

It can be easily shown that any nonsingular major of A Y_e corresponds to a tree of G. (See Seshu and Reed 10 Theorem 4-10.) It will also be shown that any nonsingular major of $A_{c, r}$ corresponds to a tree of the original graphs directed toward the reference vertex. This fact will be proved subsequently in theorem 2 but for now consider the significance of these two theorems. We now have a way of uniquely identifying all the terms in the expansion of the determiant of the node admittance matrix of a nonreciprocal network. Each term corresponds to a tree moreover, each term corresponds to a directed tree which may contain any number of bilateral elements. All this produces the desired result without the cancellations ordinarily encountered in loop and node analysis by matrix methods. Not all cancelations are avoided because each active element produces some terms which must be canceled but the ordinary cancelations resulting from passive elements are avoided thus reducing computational effort. The method improves on the Mayeda-Coates method by being more direct. That is, the determinant of the node admittance matrix is produced immediately as directed trees of a partly directed graph. The Mayeda-Coates method requires computing all trees for two graphs of approximately the some complexity, the voltage graph and the current graph, and then eliminating all but the common trees. Such a process contains needless redundant effort.

<u>Proof That a Nonsingular Submatrix of a Control Graph can be Related to a Directed Tree of a Partly Oriented Graph.</u>

We shall now formally show the importance of the control graph in determining acceptable directed trees.

Given:

1. A partly directed graph G with v vertices and e edges.

2. A tree T of the partly directed graph.

3. A control graph $G_{c, r}$ with reference vertex r and incidence matrix $A_{c, r}$ related to the partly directed graph G in the following ways:

a. r is a vertex in G.

b. All edges terminate on the reference vertex r.

c. One edge originates on every departure vertex of directed edges in G.

d. One edge originates on each of the two vertices of an undirected edge in G.

4. A subgraph $G_{ct, r}$ of $G_{c, r}$ and its incidence matrix $A_{ct, r}$ corresponding to a tree T in G.

Theorem 2: $A_{ct, r}$ is nonsingular if and only if all edges in T lie in a path directed toward r.

Proof: For the rank of $A_{ct, r}$ to be V-1 which is to say for $A_{ct, r}$ to be nonsingular, it must be possible to associate one unique controlling edge of $G_{ct, r}$ with every vertex in T. This is so because there are only V-1 vertices distinct from r and if each of these vertices were not represented uniquely, one of the following intolerable conditions would occurr:

1. The vertex would have no controlling edge associated with it. This results in a row of zeros in $A_{ct, r}$ and therefore the rank of $A_{ct, r} < V-1$ and $A_{ct, r}$ is singular.

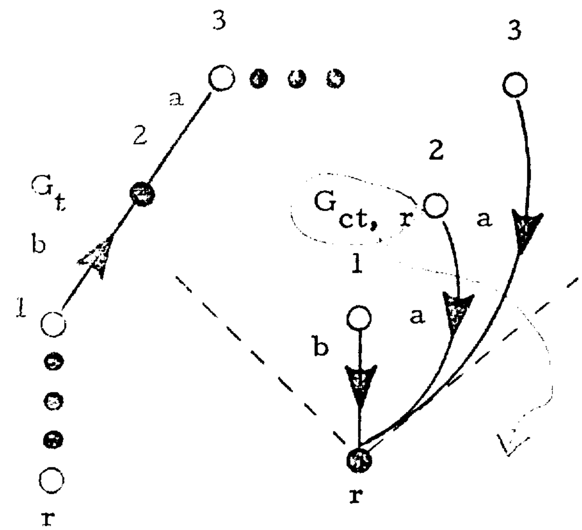
2. The vertex would have a controlling edge which is not unique. This leads to two identical rows in $A_{ct, r}$ which are linearly dependent, hence the rank $< V-1$ and $A_{ct, r}$ is singular.

Consider any directed path in T starting with an extreme vertex. If the edge attached to this vertex is oriented away from the rest of the graph, it is not possible to associate a controlling edge in $G_{ct,\ r}$ with it and condition "a" exists therefore $A_{ct,\ r}$ is singular.

If the edge attached to the extreme vertex is oriented toward the reference vertex r or if the edge is undirected, it is possible to associate a controlling edge with this vertex and $A_{ct,\ r}$ may be nonsingular.

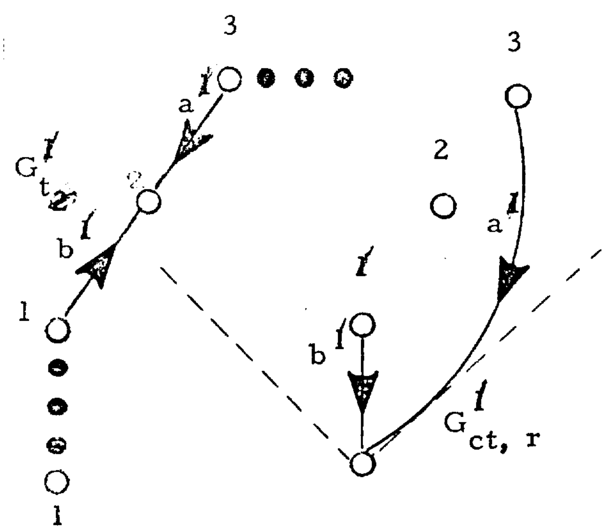
Let us continue step by step down the chosen path toward the reference vertex. We shall show with the aid of figure 15 that if we meet an edge directed away from the reference vertex, the path is not a valid directed path according to the definition 7, page and hence will not produce terms in Δ. The 2 possible ways we can encounter an edge directed away from vertex r are shown in case 1 and case 2 of figure 15. In case 1, edge a is bilateral and its control graph produces two control edges at vertices 2 and 3. That control edge between vertex 2 and the reference vertex is redundant also no other control edge is associated with this vertex and condition 2 exists. The incidence matrix of such a control graph is therefore singular. In case 2, no control edge is possible connecting vertex 2 with the reference vertex and condition 4 exists therefore the incidence matrix of the control graph is again singular.

From this it is apparent that the only acceptable condition for $A_{ct,\ r}$ to be nonsingular is for all directed edges in the path to be directed toward the reference vertex. Since the entire tree T can be decomposed into paths from extreme vertices toward the reference vertex (some edges may appear in more than one path), all directed edges in the tree must lie in a path directed toward the reference vertex if $A_{ct,\ r}$ is to be nonsingular. Conversely, if all directed edges of a tree of a partly directed graph lie in a path directed toward the



(a)

Case 1.



(b)

Case 2.

Figure 15.

A Path and Its Control Graph.

reference vertex, $G_{ct,\ r}$ the graph of controlling edges associated with that tree is nonsingular because it is possible to associate at least one unique controlling edge with each vertex.

Example of the Computation of Δ by Trees of the Partly Directed Graph.

Let us now consider a simple example of the method. Using the network of figure 16, let it be required to find the determinant of the node admittance matrix.

Our first task is to draw the partly oriented graph of the network as shown in figure 17. The directed trees of the partly directed graph are found to be:

1	2	4		2	4	5
1	3	4		-3-	-4-	-5-
2	3	4		-3-	-4-	-6-

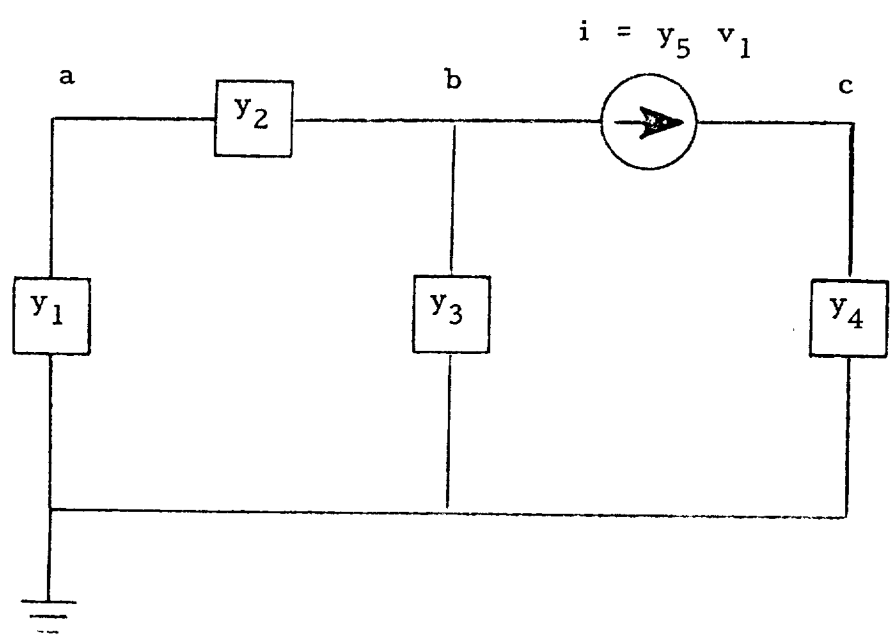


Figure 16.

Example Network.

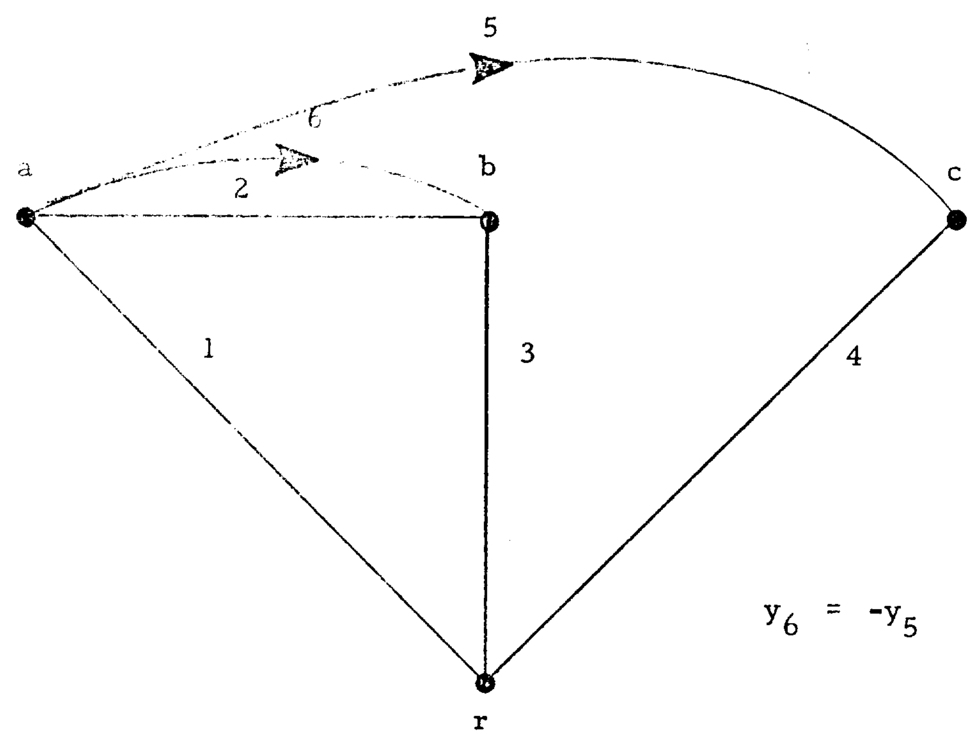


Figure 17.

Graph of the Example Network.

Because Y_6 is equal in magnitude to but opposite in sign from Y_5, the last two trees represent terms that cancel and we have:

$$\Delta = Y_1 Y_2 Y_4 + Y_1 Y_3 Y_4 + Y_2 Y_3 Y_4 + Y_2 Y_4 Y_5.$$

Determining the Admittance Matrix from the Partly Directed Graph.

It is possible to compute the node admittance matrix as follows:

Step 1.

Form the incidence matrix A of the partly directed graph in the manner shown earlier.

Step 2.

Form the incidence matrix A_d of the equivalent directed graph in the following way:

For every column in A in which two +1's appear, change one sign to -1 for either entry. This corresponds to an arbitrary assignment of edge orientation for bilateral elements. The resultant matrix A_d could have been obtained from a directed graph equivalent; however, this process eliminates the need for redrawing the graph.

Step 3.

Form the unilateral edge selection matrix A' from the partly directed graph in the following way:

$$A' = \left[a'_{ij} \right]$$

where

$a'_{ij} = a_{ij}$ for bilateral edges.

$a_{ij} = 1$ for any unilateral edge j departing from vertex i.

$a'_{ij} = 0$ otherwise.

Step 4.

Form the diagonal edge admittance matrix Y_e including all admittance elements as follows:

$Y_e = \left[Y_{ejj} \right]$ where y_{ejj} is the admittance of each edge j, and is the (j, j)th element of Y_e.

Step 5.

The node admittance matrix is now found according to the following formula:

$$Y_n = A_d \, Y_e \, A'^t .$$

For our example,

$$A_d = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$Y_e = \begin{bmatrix} y_1 & & & 0 & & \\ & y_2 & & & & \\ & & y_3 & & & \\ & & & y_4 & & \\ & & & & y_5 & \\ & & 0 & & & y_6 \end{bmatrix}$$

and,

$$Y_n = \begin{bmatrix} y_1 & y_2 & 0 & 0 & y_5 & -y_5 \\ 0 & -y_2 & y_3 & 0 & 0 & +y_5 \\ 0 & 0 & 0 & y_4 & -y_5 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$Y_n = \begin{bmatrix} (y_1 + y_2) & -y_2 & 0 \\ (-y_2 + y_5) & (y_2 + y_3) & 0 \\ -y_5 & 0 & y_4 \end{bmatrix} .$$

Computing Cofactors from the Partly Oriented Graph.

In the previous sections, we have seen how the node admittance matrix and its determinant can be derived from the partly directed graph. The process of computing cofactors is more complex but still follows the same basic idea, that is, the algebraic expressions representing the incidence matrices of a graph are combined with the edge admittance matrix in such a way as to form the desired cofactors. Then, it is shown how these expressions bear a one-to-one correspondence with the directed 2-trees of the partly directed graph. Starting with the definition of a cofactor, we have:

$$\Delta_{ij} = (-1)^{i+j} M_{ij}$$

where Δ_{ij} = the (i, j)th cofactor of det Y_n.

M_{ij} = the (i, j)th major of det Y_n.

and M_{ij} can be related to the incidence matrices of the original network as follows:

$$M_{ij} = A_{-i} \; Y_e \; A^t_{(c,\, r)-j}$$

where

A_{-i} = the incidence matrix of G with row i deleted

Y_e = the edge admittance matrix

$A_{(c,\, r)-j}$ = the incidence matrix of the control graph $G_{c,\, r}$ with row j deleted.

Again, by the Binet-Cauchy Theorem, we can relate M_{ij} to the product of corresponding nonsingular majors of A_{-i} Y_e and $A_{(c,\, r)}{}^t{}_{-j}$.

$$M_{ij} = \sum \left(\begin{array}{l} \text{Products of corresponding nonsingular} \\ \text{majors of } A_{-i}\; Y_e \text{ and } A_{(c,\, r)}{}^t{}_{-j} \end{array} \right).$$

Proceeding one step further, we know that all nonsingular majors of $A_{-i} \; Y_e$ correspond to 2-trees of the original graph G with vertex i in one part and the reference vertex in the other part. We can arrive at this conclusion by investigating the topological significance of $A_{-i} \; Y_e$. The deletion of row i can be directly related to the removal of vertex i in G to form G_{-i}. This is done by identifying this vertex with the reference vertex. Now, all nonsingular majors of $A_{-i} \; Y_e$ correspond to the tree of G_{-i} or the 2-trees of G. (See Seshu and Reed Theorems 4-10 and 7-3 [10].) Altogether, we have shown nothing very new or spectacular but the next step is new and represents a significant departure from the conventional technique.

It can be shown that the nonsingular majors of $A_{(c,\ r)-j}$ corresponding to the trees of G_{-i} have a one-to-one relationship with the directed 2-trees of the partly directed graph with vertices i and j in one part and j being the reference vertex for that part also vertex r in the other part and remaining the reference vertex for that part. It can be seen from this that every directed 2-tree of the partly oriented graph has two reference vertices.

The proof for this depends on theorem 2 in the following way:

$A_{(c,\ r)-j}$ is the incidence matrix of the control graph for G_{-j} the modified graph where vertex j is identified with the reference vertex. Since the only majors of $A_{(c,\ r)-j}$ that will produce a nonzero result correspond to a tree G_{-i}, we need only consider majors from a matrix of the following type: $A_{(ct,\ r)-j}$. By theorem 2, we know that all nonsingular majors of this type correspond to directed trees of G_{-j} which can be interpreted as directed 2-trees of G with vertex j in one part and the reference vertex r in the other part. Since these 2-trees must also correspond to a tree of G_{-i}, or 2-trees with vertex i in one part and the reference vertex in the other part, each must contain both vertices i and j in one part and the reference vertex in the other. This can be expressed in the algebra of sets as follows:

$$T_2 = T_{2_{i,\, r}} \bigcap T_{2_{j,\, r}}$$

where

T_2	=	the set of 2-trees which we seek.
$T_{2_{i,\, r}}$	=	the set of 2-trees with vertex i in one part and vertex r in the other.

We can divide these set of 2-trees into two parts each: namely those which contain vertices i and j in one part and those which do not. This is so because all vertices including i and j must be represented in one part or the other part of a 2-tree set. Clarifying this identity we have for example:

$$T_{2_{i,\, r}} = T_{2_{ij,\, r}} + T_{2_{i,\, jr}}$$

which says in words:

The set of 2-trees with vertex i in one part and vertex r in the other.	=	The subset of 2-trees with vertices i and j in one part and vertex r in the other.	+	The subset of 2-trees with vertex i in one part and vertices j and r in the other.

also,

$$T_{2_{j,\, r}} = T_{2_{ij,\, r}} + T_{2_{j,\, ir}}$$

and it follows by taking the intersection of these two sets:

$$T_2 = (T_{2_{ij,\, r}} + T_{2_{i,\, jr}}) \cap (T_{2_{ij,\, r}} + T_{2_{j,\, ir}})$$

$$T_2 = T_{2_{ij,\, r}}.$$

We know now what classes of 2-trees represent the nonsingular majors of Y_n and now we are free to explore the topological significance of these. Recalling that the 2-trees were originally found by identifying a vertex j with the reference vertex in G_{-j}; further recalling that they correspond to trees of G_{-j}, we depend on theorem 2 which deals with only directed trees with vertex r as the reference. By understanding how G_{-j} was developed, we can now reverse the process and extract the jth vertex from the reference vertex to see the significance of the sense of direction for directed paths in each of the two parts of the two-tree. All valid parts of the 2-tree are in themselves directed trees terminating either on vertex j or vertex r and we have:

$$\Delta_{ij} = (-1)^{i+j} \; W_{ij,\, r}$$

where $W_{ij,\, r}$ = the sum of all 2-tree admittance products with i and j in one part, j serving as reference for that part; and r in the other part also serving as a reference for that part.

<u>Example of the Computation of Cofactors.</u>

As in illustration of the method, let us consider the entire set of cofactors for the example of figure 16. The directed 2-trees of the graph G with respect to all cofactors are given in table 2.

Table 2.

Directed 2-Trees of the Example Network.

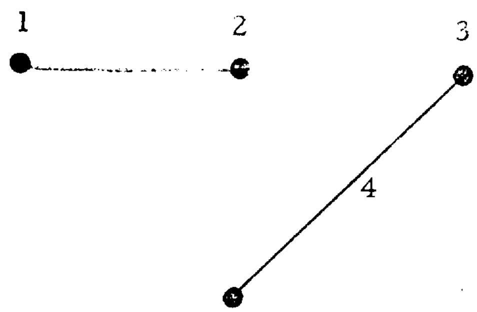
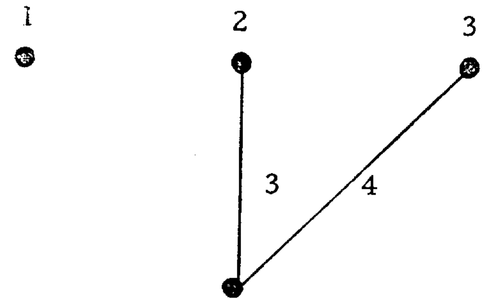
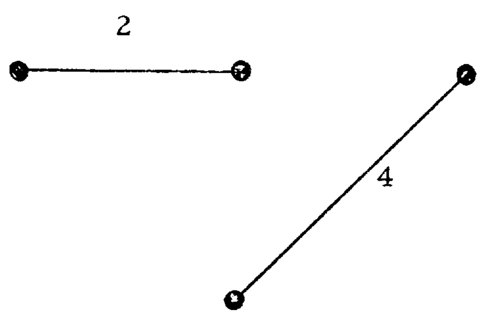
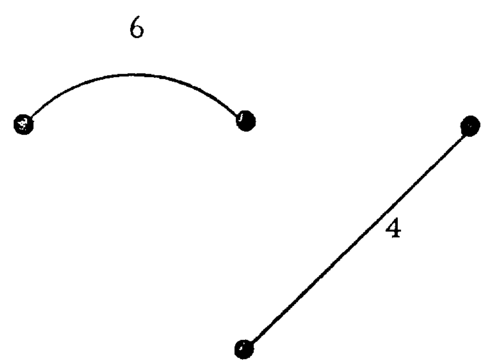
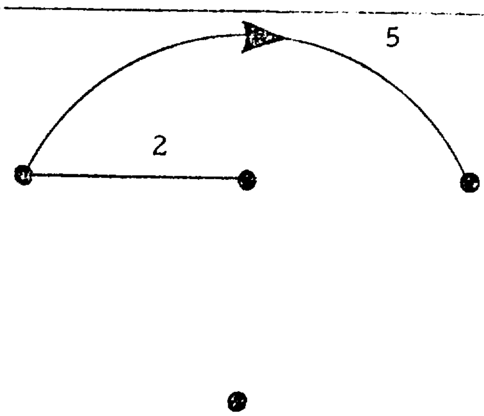
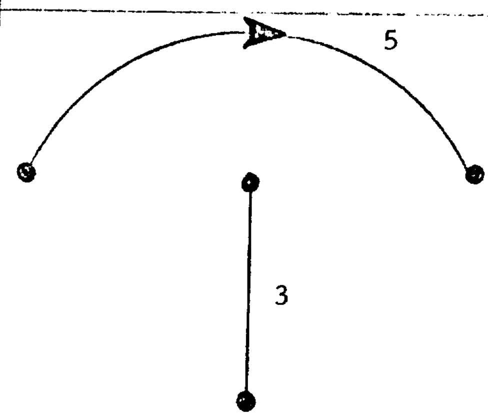
$W_{11,\ r}$	+	 $y_2 \quad y_4$	 $y_3 \quad y_4$
$W_{12,\ r}$	-	 $-y_2 \quad y_4$	 $-y_4 \quad y_6$
$W_{13,\ r}$	+	 $y_2 \quad y_5$	 $y_3 \quad y_5$

Table 2 (Continued).

Directed 2-Trees of the Example Network.

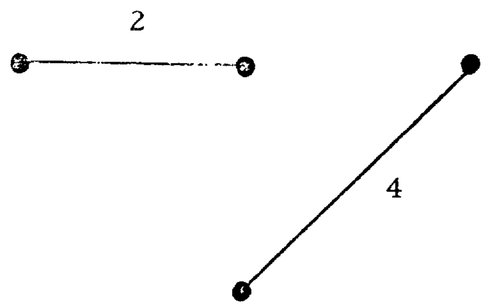
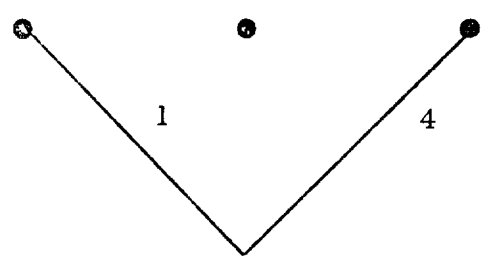
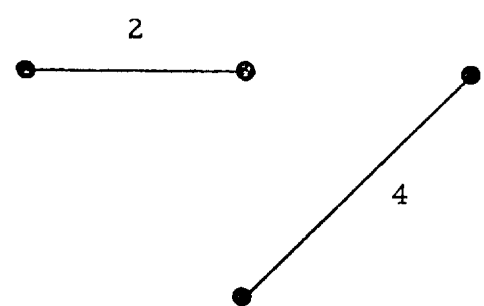
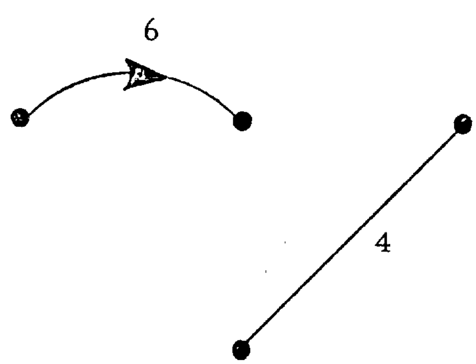
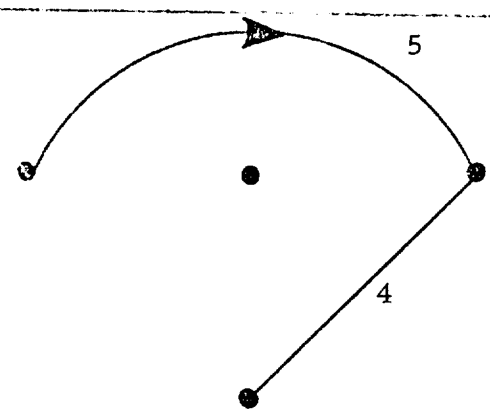
$W_{21, r}$	-	 $-y_2 \; y_4$	
$W_{22, r}$	+	 $y_1 \; y_4$	 $y_2 \; y_4$
$W_{22, r}$ (Con't)	+	 $y_4 \; y_6$	 $y_4 \; y_5$

Table 2 (Continued).

Directed 2-Trees of the Example Network.

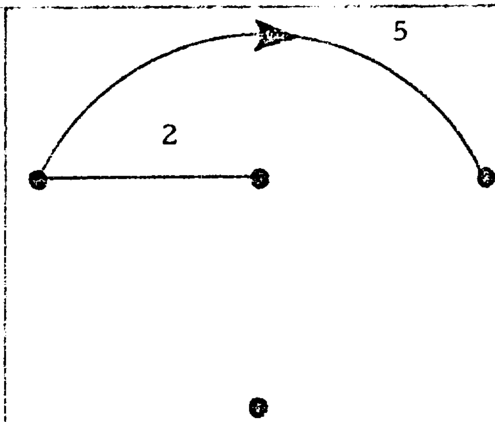
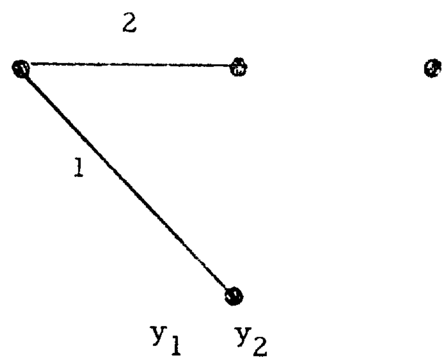
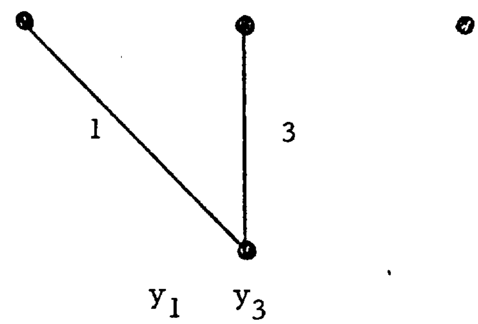
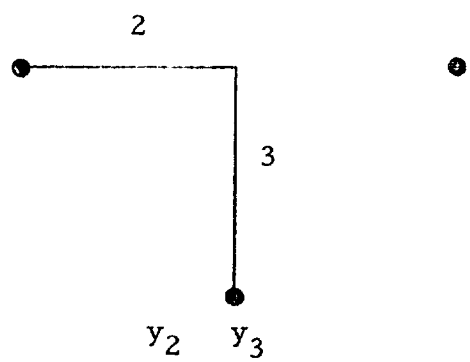
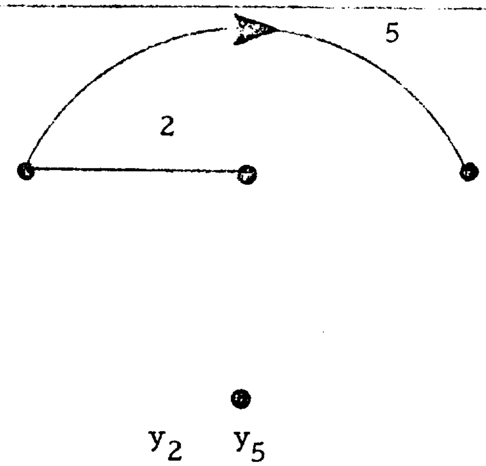
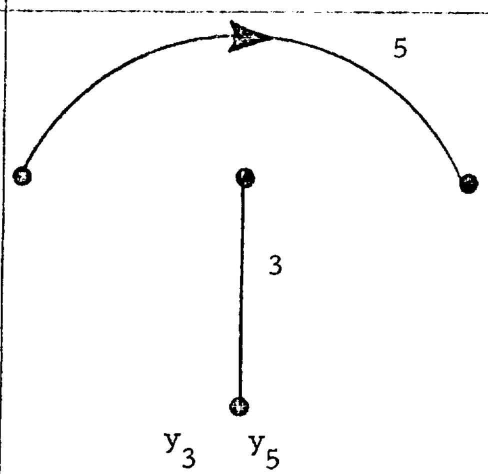
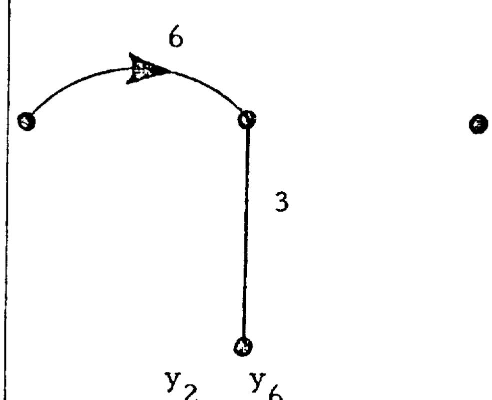
$W_{23,\ r}$	-	 $-y_2 \quad y_5$
$W_{31,\ r}$	+	None
$W_{32,\ r}$	-	None

Table 2 (Continued).

Directed 2-Trees of the Example Network.

$W_{33,\ r}$	+		
$W_{33,\ r}$ (Con't)	+		
$W_{33,\ r}$ (Con't)	+		

and,

$$\Delta_{11} = Y_2 Y_4 + Y_3 Y_4.$$

$$\Delta_{12} = Y_2 Y_4 - Y_4 Y_6.$$

$$\Delta_{13} = Y_2 Y_5 + Y_3 Y_5.$$

$$\Delta_{21} = - Y_2 Y_4.$$

$$\Delta_{22} = Y_1 Y_4 + Y_2 Y_4 + Y_4 Y_5 + Y_4 Y_6.$$

$$\Delta_{23} = - Y_2 Y_5.$$

$$\Delta_{31} = 0.$$

$$\Delta_{32} = 0.$$

$$\Delta_{33} = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 + Y_2 Y_5 + Y_3 Y_5 + Y_3 Y_5 + Y_3 Y_6.$$

From this it is clear that no sign problem exists and all cofactors are easily determined. It relates directly to the process of determining 2-trees of a reciprocal network with only one exception that directed trees be considered with respect to reference vertices as discussed above. The literature for finding 2-trees by computer abounds and we now have an easy method of applying the benefits of this work to non-reciprocal networks.

Conclusions.

Thus far, we have reviewed the current methods of topological analysis of general networks and have proposed a new method. This new method has the advantage of being the simplest in the sense that it is an exclusively topological method where it used the simplest topological graph structure possible and the simplest process for determing the terms in the expansion of the determinant and cofactors of the mode admittance matrix. The method bears a strong relationship to topological analysis of reciprocal networks and is very well suited to implementation by digital computer.

All network functions can be related to the determinant of the node admittance matrix and its cofactors. Therefore, the development shown here including solutions to these problems in terms of trees and 2 trees is applicable. However, some network functions are more conveniently represented by 3-trees , [24] , [25] . A natural extension to this new method would be to show the usefulness and significance of sets of k-trees of the partly directed graph.

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